

Finite-dimensional irreducible modules for the three-point \mathfrak{sl}_2 loop algebra

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Abstract

Recently Brian Hartwig and the second author found a presentation for the three-point \mathfrak{sl}_2 loop algebra by generators and relations. To obtain this presentation they defined a Lie algebra \boxtimes by generators and relations, and displayed an isomorphism from \boxtimes to the three-point \mathfrak{sl}_2 loop algebra. In this paper we describe the finite-dimensional irreducible \boxtimes -modules from multiple points of view.

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1 Introduction

In [16] Hartwig and the second author found a presentation for the three-point \mathfrak{sl}_2 loop algebra via generators and relations. To obtain this presentation they defined a Lie algebra \boxtimes (pronounced “tet”) by generators and relations, and displayed an isomorphism from \boxtimes to the three-point \mathfrak{sl}_2 loop algebra. \boxtimes has essentially six generators, and it is natural to identify these with the six edges of a tetrahedron. The action of the symmetric group S_4 on the tetrahedron induces an action of S_4 on \boxtimes as a group of automorphisms [16, Section 2]. For each face of the tetrahedron the three surrounding edges form a basis for a subalgebra of \boxtimes that is isomorphic to \mathfrak{sl}_2 [16, Corollary 12.4]. Any five of the six edges of the tetrahedron generate a subalgebra of \boxtimes that is isomorphic to the \mathfrak{sl}_2 loop algebra [16, Corollary 12.6]. Each pair of opposite edges of the tetrahedron generate a subalgebra of \boxtimes that is isomorphic to the Onsager algebra \mathcal{O} [16, Corollary 12.5]; let us call these Onsager subalgebras. Then \boxtimes is the direct sum of its three Onsager subalgebras [16, Theorem 11.6]. In [13] Elduque found an attractive decomposition of \boxtimes into a direct sum of three abelian subalgebras, and he showed how these subalgebras are related to the Onsager subalgebras. In [31] Pascasio and the second author gave an action of \boxtimes on the standard module for each Hamming

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association scheme. In [4] Bremner obtained the universal central extension of the three-point \mathfrak{sl}_2 loop algebra. By modifying the defining relations for \boxtimes , Benkart and the second author obtained a presentation for this extension by generators and relations [2]. In [15] Hartwig classified the finite-dimensional irreducible \boxtimes -modules over an algebraically closed field \mathbb{F} with characteristic 0. He did this by displaying a bijection between (i) the set of isomorphism classes of finite-dimensional irreducible \boxtimes -modules; (ii) the set of isomorphism classes of finite-dimensional irreducible \mathcal{O} -modules that have type $(0, 0)$ [15, Theorems 1.7, 1.8]. The modules in (ii) were classified earlier by Davies [10], [11]; see also Date and Roan [9]. A summary of this classification is given in [15, Theorems 1.3, 1.4, 1.6].

In this paper we consider further the finite-dimensional irreducible \boxtimes -modules. We pick up where Davies, Date and Roan, and Hartwig left off, and obtain a description of these modules that has more detail and takes a different point of view. Our results are summarized as follows. Among the finite-dimensional irreducible \boxtimes -modules we identify a special case called an evaluation module. Each evaluation module is written $V_d(a)$, where $d \geq 1$ is an integer and $a \in \mathbb{F} \setminus \{0, 1\}$. The scalar a is called the evaluation parameter. To get the \boxtimes -module $V_d(a)$ we start with the irreducible \mathfrak{sl}_2 -module V_d of dimension $d + 1$, and pull back the \mathfrak{sl}_2 -module structure via an evaluation homomorphism $EV_a : \boxtimes \rightarrow \mathfrak{sl}_2$. We note that our evaluation homomorphism is different from the one in [9]. We show that every finite-dimensional irreducible \boxtimes -module is a tensor product of evaluation modules. We give three characterizations that enable us to recognize the evaluation modules among all finite-dimensional irreducible \boxtimes -modules. Upon twisting an evaluation module via an element of S_4 we get an evaluation module whose evaluation parameter is potentially different. We describe how twisting affects the evaluation parameter. To do this, we first display an action of S_4 on $\mathbb{F} \setminus \{0, 1\}$ by linear fractional transformations. We then show that for an evaluation module $V_d(a)$ and $\sigma \in S_4$, the \boxtimes -module $V_d(a)$ twisted via σ is isomorphic to $V_d(\sigma(a))$. Let G denote the kernel of the above S_4 -action on $\mathbb{F} \setminus \{0, 1\}$. We show that G is the unique normal subgroup of S_4 that has cardinality 4. For a given evaluation module $V_d(a)$ we consider 24 bases that are described as follows. Let \mathbb{I} denote the vertex set of the tetrahedron. For mutually distinct $i, j, k, \ell \in \mathbb{I}$ we define an (ordered) basis $[i, j, k, \ell]$ of $V_d(a)$ such that (i) the basis diagonalizes the \boxtimes -generator identified with the edge of the tetrahedron incident with k and ℓ ; (ii) the sum of the basis vectors is a common eigenvector for the three \boxtimes -generators identified with the edges incident with i . We find the matrices that represent all the \boxtimes -generators with respect to $[i, j, k, \ell]$. Moreover we find the transition matrices from the basis $[i, j, k, \ell]$ to the bases

$$[j, i, k, \ell], \quad [i, k, j, \ell], \quad [i, j, \ell, k].$$

The first matrix is diagonal, the second matrix is lower triangular, and the third matrix is the identity reflected about a vertical axis. We wind up our discussion of evaluation modules by showing how they can be concretely realized using homogeneous polynomials in two variables. Now consider a general finite-dimensional irreducible \boxtimes -module V . For $\sigma \in G$ we show that V twisted via σ is isomorphic to V . We associate with V two polynomials whose algebraic structure reflects that of V . The first polynomial, denoted S_V and called the shape polynomial, has coefficients that encode the eigenspace dimensions for the action of the \boxtimes -generators on V . We show how a certain factorization of S_V over the integers

corresponds to the factorization of V into a tensor product of evaluation modules. The second polynomial, denoted P_V and called the Drinfel'd polynomial, is analogous to the Drinfel'd polynomial for the classical \mathfrak{sl}_2 loop algebra [5], [6], [26]. We show that the factorization of P_V into linear factors corresponds to the factorization of V into a tensor product of evaluation modules. Moreover we show that the map $V \mapsto P_V$ induces a bijection between (i) the set of isomorphism classes of finite-dimensional irreducible \boxtimes -modules; (ii) the set of univariable polynomials over \mathbb{F} that have constant coefficient 1 and do not vanish at 1. We display a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on V such that

$$\langle \xi.u, v \rangle = -\langle u, \xi.v \rangle \quad \xi \in \boxtimes, \quad u, v \in V.$$

We use this form to show that the \boxtimes -module V is isomorphic to the dual \boxtimes -module V^* . There is another type of bilinear form on V that is of interest. For a nonidentity $\sigma \in G$ we display a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_\sigma$ on V such that

$$\langle \xi.u, v \rangle_\sigma = -\langle u, \sigma(\xi).v \rangle_\sigma \quad \xi \in \boxtimes, \quad u, v \in V.$$

We remark on our motivation to investigate \boxtimes . This has to do with the tridiagonal pairs [19], [20], [21], [22] and the closely related Leonard pairs [27], [28], [29], [35], [36], [37], [39], [40]. A Leonard pair is a pair of semisimple linear transformations on a finite-dimensional vector space, each of which acts in an irreducible tridiagonal fashion on an eigenbasis for the other [35, Definition 1.1]. There is a close connection between the Leonard pairs and the orthogonal polynomials that make up the terminating branch of the Askey scheme [25], [35], [38]. A tridiagonal pair is a mild generalization of a Leonard pair [19, Definition 1.1]. A pair of \boxtimes -generators identified with opposite edges of the tetrahedron act on each finite-dimensional irreducible \boxtimes -module as a tridiagonal pair [15, Theorem 1.7, Corollary 2.7]. For the details on this connection see [15, Section 2].

2 The three-point \mathfrak{sl}_2 loop algebra

In this section we recall the three-point \mathfrak{sl}_2 loop algebra and its relationship to \boxtimes .

Throughout the paper \mathbb{F} denotes an algebraically closed field with characteristic 0. By \mathfrak{sl}_2 we mean the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$.

Definition 2.1 Let t denote an indeterminate and let $\mathbb{F}[t, t^{-1}, (t-1)^{-1}]$ denote the \mathbb{F} -algebra of all Laurent polynomials in $t, t-1$ that have coefficients in \mathbb{F} . We abbreviate

$$\mathcal{A} = \mathbb{F}[t, t^{-1}, (t-1)^{-1}].$$

We consider the Lie algebra over \mathbb{F} consisting of the \mathbb{F} -vector space

$$\mathfrak{sl}_2 \otimes \mathcal{A}, \quad \otimes = \otimes_{\mathbb{F}}$$

and Lie bracket

$$[u \otimes a, v \otimes b] = [u, v] \otimes ab, \quad u, v \in \mathfrak{sl}_2, \quad a, b \in \mathcal{A}.$$

We call $\mathfrak{sl}_2 \otimes \mathcal{A}$ the *three-point \mathfrak{sl}_2 loop algebra*.

See [4], [14], [33], [34] for information on multipoint loop algebras and related topics.

We now recall the Lie algebra \boxtimes .

Definition 2.2 [16, Definition 1.1] Let \boxtimes denote the Lie algebra over \mathbb{F} that has generators

$$\{x_{ij} \mid i, j \in \mathbb{I}, i \neq j\} \quad \mathbb{I} = \{0, 1, 2, 3\} \quad (1)$$

and the following relations:

(i) For distinct $i, j \in \mathbb{I}$,

$$x_{ij} + x_{ji} = 0.$$

(ii) For mutually distinct $i, j, k \in \mathbb{I}$,

$$[x_{ij}, x_{jk}] = 2x_{ij} + 2x_{jk}.$$

(iii) For mutually distinct $i, j, k, \ell \in \mathbb{I}$,

$$[x_{ij}, [x_{ij}, [x_{ij}, x_{k\ell}]] = 4[x_{ij}, x_{k\ell}]. \quad (2)$$

We call \boxtimes the *tetrahedron algebra* or “tet” for short.

Note 2.3 The relations (2) are known as the *Dolan-Grady* relations [1], [9], [10], [11], [12], [24].

After a comment we will display a Lie algebra isomorphism from \boxtimes to $\mathfrak{sl}_2 \otimes \mathcal{A}$.

Recall that the Lie algebra \mathfrak{sl}_2 has a basis e, f, h and

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Following [16, Lemma 3.2] we define

$$x = 2e - h, \quad y = -2f - h, \quad z = h. \quad (3)$$

Then x, y, z is a basis for \mathfrak{sl}_2 and

$$[x, y] = 2x + 2y, \quad [y, z] = 2y + 2z, \quad [z, x] = 2z + 2x.$$

By analogy with [23], [41] we call x, y, z the *equitable basis* for \mathfrak{sl}_2 . See [3] for a detailed study of this basis.

Proposition 2.4 [16, Proposition 6.5] *There exists an isomorphism of Lie algebras $\psi : \boxtimes \rightarrow \mathfrak{sl}_2 \otimes \mathcal{A}$ that sends*

$$\begin{aligned} x_{12} &\mapsto x \otimes 1, & x_{03} &\mapsto y \otimes t + z \otimes (t - 1), \\ x_{23} &\mapsto y \otimes 1, & x_{01} &\mapsto z \otimes (1 - t^{-1}) - x \otimes t^{-1}, \\ x_{31} &\mapsto z \otimes 1, & x_{02} &\mapsto x \otimes (1 - t)^{-1} + y \otimes t(1 - t)^{-1}, \end{aligned}$$

where x, y, z is the equitable basis for \mathfrak{sl}_2 .

We mention a result for later use.

Lemma 2.5 [16, Corollary 12.1] *For mutually distinct $i, j, k \in \mathbb{I}$ there exists an injective homomorphism of Lie algebras from \mathfrak{sl}_2 to \boxtimes that sends*

$$x \mapsto x_{ij}, \quad y \mapsto x_{jk}, \quad z \mapsto x_{ki}.$$

3 Finite-dimensional irreducible \boxtimes -modules

In this section we review some basic facts and notation concerning finite-dimensional irreducible \boxtimes -modules. This material is summarized from [15].

Let V denote a vector space over \mathbb{F} with finite positive dimension. Let $\{s_n\}_{n=0}^d$ denote a sequence of positive integers whose sum is the dimension of V . By a *decomposition of V of shape $\{s_n\}_{n=0}^d$* we mean a sequence $\{V_n\}_{n=0}^d$ of subspaces of V such that V_n has dimension s_n for $0 \leq n \leq d$ and $V = \sum_{n=0}^d V_n$ (direct sum). We call V_n the *n th component* of the decomposition. We call d the *diameter* of the decomposition. For notational convenience we define $V_{-1} = 0$ and $V_{d+1} = 0$. By the *inversion* of $\{V_n\}_{n=0}^d$ we mean the decomposition $\{V_{d-n}\}_{n=0}^d$.

Now let V denote a finite-dimensional irreducible \boxtimes -module. By [15, Theorem 3.8] each generator x_{ij} of \boxtimes is semisimple on V . Moreover there exists an integer $d \geq 0$ such that for each generator x_{ij} the set of distinct eigenvalues on V is $\{d - 2n \mid 0 \leq n \leq d\}$. We call d the *diameter* of V . For distinct $i, j \in \mathbb{I}$ we define a decomposition of V called $[i, j]$. The decomposition $[i, j]$ has diameter d . For $0 \leq n \leq d$ the n th component of $[i, j]$ is the eigenspace of x_{ij} for the eigenvalue $2n - d$. Using Definition 2.2(i) we find $[j, i]$ is the inversion of $[i, j]$. By [15, Corollary 3.6], for distinct $i, j \in \mathbb{I}$ the shape of $[i, j]$ is independent of i, j . We denote this shape by $\{\rho_n\}_{n=0}^d$ and note that $\rho_n = \rho_{d-n}$ for $0 \leq n \leq d$. By the *shape of V* we mean the sequence $\{\rho_n\}_{n=0}^d$.

Let V denote a finite-dimensional irreducible \boxtimes -module. For distinct $i, j \in \mathbb{I}$ and for distinct $r, s \in \mathbb{I}$ we now describe the action of x_{rs} on the decomposition $[i, j]$ of V . Denote this decomposition by $\{V_n\}_{n=0}^d$. Then by [15, Theorem 3.3], for $0 \leq n \leq d$ the action of x_{rs} on V_n is given in the table below.

| Case | Action of x_{rs} on V_n |
|-------------------------|---|
| $r = i, \quad s = j$ | $(x_{rs} - (2n - d)I)V_n = 0$ |
| $r = j, \quad s = i$ | $(x_{rs} - (d - 2n)I)V_n = 0$ |
| $r = j, \quad s \neq i$ | $(x_{rs} - (d - 2n)I)V_n \subseteq V_{n+1}$ |
| $r \neq i, \quad s = j$ | $(x_{rs} - (2n - d)I)V_n \subseteq V_{n+1}$ |
| $r = i, \quad s \neq j$ | $(x_{rs} - (2n - d)I)V_n \subseteq V_{n-1}$ |
| $r \neq j, \quad s = i$ | $(x_{rs} - (d - 2n)I)V_n \subseteq V_{n-1}$ |
| i, j, r, s distinct | $x_{rs}V_n \subseteq V_{n-1} + V_n + V_{n+1}$ |

We recall the notion of a *flag*. For the moment let V denote a vector space over \mathbb{F} with finite positive dimension and let $\{s_n\}_{n=0}^d$ denote a sequence of positive integers whose sum is the dimension of V . By a *flag on V of shape $\{s_n\}_{n=0}^d$* we mean a nested sequence $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d$ of subspaces of V such that the dimension of U_n is $s_0 + \cdots + s_n$ for $0 \leq n \leq d$. We call U_n the *n th component* of the flag. We call d the *diameter* of the flag. We observe that $U_d = V$. The following construction yields a flag on V . Let $\{V_n\}_{n=0}^d$ denote a decomposition of V of shape $\{s_n\}_{n=0}^d$. Define

$$U_n = V_0 + V_1 + \cdots + V_n \quad (0 \leq n \leq d).$$

Then the sequence $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d$ is a flag on V of shape $\{s_n\}_{n=0}^d$. We say this flag is *induced* by the decomposition $\{V_n\}_{n=0}^d$. We now recall what it means for two flags to

be *opposite*. Suppose we are given two flags on V with the same diameter: $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d$ and $U'_0 \subseteq U'_1 \subseteq \cdots \subseteq U'_d$. We say these flags are *opposite* whenever there exists a decomposition $\{V_n\}_{n=0}^d$ of V such that

$$U_n = V_0 + V_1 + \cdots + V_n, \quad U'_n = V_d + V_{d-1} + \cdots + V_{d-n}$$

for $0 \leq n \leq d$. In this case

$$U_i \cap U'_j = 0 \quad \text{if } i + j < d \quad (0 \leq i, j \leq d) \quad (4)$$

and

$$V_n = U_n \cap U'_{d-n} \quad (0 \leq n \leq d). \quad (5)$$

In particular the decomposition $\{V_n\}_{n=0}^d$ is uniquely determined by the given flags.

We now return our attention to \boxtimes -modules. Let V denote a finite-dimensional irreducible \boxtimes -module of diameter d . By [15, Lemma 5.3] there exists a collection of flags on V , denoted $[i], i \in \mathbb{I}$, such that for distinct $i, j \in \mathbb{I}$ the decomposition $[i, j]$ induces $[i]$. By construction, for $i \in \mathbb{I}$ the shape of the flag $[i]$ coincides with the shape of V . By construction, the flags $[i], i \in \mathbb{I}$ are mutually opposite. Also by construction, for distinct $i, j \in \mathbb{I}$ and for $0 \leq n \leq d$ the n th component of $[i, j]$ is equal to the intersection of the following two sets:

- (i) component n of the flag $[i]$;
- (ii) component $d - n$ of the flag $[j]$.

4 Evaluation modules for \boxtimes ; preliminaries

In the next section we will define a class of \boxtimes -modules called evaluation modules. In this section we make some preliminary comments to set the stage.

Lemma 4.1 *Let K denote an ideal of \boxtimes such that $K \neq \boxtimes$. Then for mutually distinct $i, j, k \in \mathbb{I}$ the following sum is direct:*

$$\mathbb{F}x_{ij} + \mathbb{F}x_{jk} + \mathbb{F}x_{ki} + K. \quad (6)$$

Proof: We first claim that $x_{rs} \notin K$ for all distinct $r, s \in \mathbb{I}$. To prove the claim it suffices to show that $x_{rs} \in K$ implies $x_{su} \in K$ for mutually distinct $r, s, u \in \mathbb{I}$. If $x_{rs} \in K$ then $[x_{rs}, x_{su}] \in K$ since K is an ideal. Also $[x_{rs}, x_{su}] = 2x_{rs} + 2x_{su}$ by Definition 2.2(ii) so $x_{su} \in K$ and the claim follows. We can now easily show that the sum (6) is direct. By Lemma 2.5 the generators x_{ij}, x_{jk}, x_{ki} form a basis for a subalgebra L of \boxtimes that is isomorphic to \mathfrak{sl}_2 . Recall that \mathfrak{sl}_2 is simple so L is simple. By this and since $K \cap L$ is an ideal of L we find $K \cap L = 0$ or $L \subseteq K$. The second possibility cannot occur by the claim, so $K \cap L = 0$ and this means that the sum (6) is direct. \square

Lemma 4.2 *For a nonzero Lie algebra homomorphism $\boxtimes \rightarrow \mathfrak{sl}_2$ and for mutually distinct $i, j, k \in \mathbb{I}$ the images of x_{ij}, x_{jk}, x_{ki} form a basis for \mathfrak{sl}_2 .*

Proof: Let K denote the kernel of the homomorphism and note that K is an ideal of \boxtimes . The homomorphism is nonzero so $K \neq \boxtimes$. Applying Lemma 4.1 to K we find that the images of x_{ij}, x_{jk}, x_{ki} are linearly independent in \mathfrak{sl}_2 . The result follows since \mathfrak{sl}_2 has dimension 3. \square

Proposition 4.3 *For a nonzero Lie algebra homomorphism $\boxtimes \rightarrow \mathfrak{sl}_2$ there exists a unique $a \in \mathbb{F}$ such that the following are in the kernel:*

$$\begin{aligned} ax_{01} + (1-a)x_{02} - x_{03}, & \quad ax_{10} + (1-a)x_{13} - x_{12}, \\ ax_{23} + (1-a)x_{20} - x_{21}, & \quad ax_{32} + (1-a)x_{31} - x_{30}. \end{aligned}$$

Moreover $a \neq 0$ and $a \neq 1$.

Proof: We first claim that for mutually distinct $i, j, k, \ell \in \mathbb{I}$ the images of

$$x_{ij} - x_{ik}, \quad x_{ik} - x_{i\ell}$$

in \mathfrak{sl}_2 are nonzero and linearly dependent. These images are nonzero by Definition 2.2(i) and Lemma 4.2. To show that they are linearly dependent, let K denote the kernel of the homomorphism. By Lemma 4.2 there exist scalars $\alpha, \beta, \gamma \in \mathbb{F}$ such that

$$x_{i\ell} - \alpha x_{ij} - \beta x_{jk} - \gamma x_{ki} \tag{7}$$

is in K . Taking the Lie bracket of (7) with x_{ij} and simplifying the result using Definition 2.2(i),(ii) we find

$$x_{i\ell} + (\beta - \gamma - 1)x_{ij} + \beta x_{jk} - \gamma x_{ki} \tag{8}$$

is in K . Comparing (7), (8) and using Lemma 4.2 we find $\beta = 0$ and $\gamma = \alpha - 1$. Evaluating (7) using this and Definition 2.2(i) we find that K contains

$$\alpha(x_{ij} - x_{ik}) + x_{ik} - x_{i\ell}$$

and the claim follows. We now apply the claim for the following values of (i, j, k, ℓ) :

$$(0, 1, 2, 3), \quad (1, 0, 3, 2), \quad (2, 3, 0, 1), \quad (3, 2, 1, 0).$$

We find that there exist unique scalars $a, b, c, d \in \mathbb{F}$ such that each of

$$ax_{01} + (1-a)x_{02} - x_{03}, \tag{9}$$

$$bx_{10} + (1-b)x_{13} - x_{12}, \tag{10}$$

$$cx_{23} + (1-c)x_{20} - x_{21}, \tag{11}$$

$$dx_{32} + (1-d)x_{31} - x_{30} \tag{12}$$

is in K . None of a, b, c, d is equal to 0 or 1 in view of Lemma 4.2 and Definition 2.2(i). Define $E \in \boxtimes$ to be (9) plus a/b times (10) plus $(1-a)/(1-c)$ times (11) plus (12). By

construction $E \in K$. Also by Definition 2.2(i), E is equal to a certain linear combination of x_{12}, x_{23}, x_{31} . In this linear combination each coefficient must be 0 since the images of x_{12}, x_{23}, x_{31} are linearly independent in \mathfrak{sl}_2 . Therefore

$$\frac{a}{b}, \quad \frac{1-a}{1-c}, \quad \frac{d}{c}, \quad \frac{1-d}{1-b}$$

coincide and this yields $a = b = c = d$ after a brief calculation. The result follows. \square

We finish this section with a comment. Up to isomorphism there exists a unique \boxtimes -module with dimension 1, and every element of \boxtimes is 0 on this \boxtimes -module. We call this \boxtimes -module *trivial*.

Lemma 4.4 *Let V denote a nontrivial finite-dimensional irreducible \boxtimes -module. Then for mutually distinct $i, j, k \in \mathbb{I}$ the actions of x_{ij}, x_{jk}, x_{ki} on V are linearly independent.*

Proof: Let $K = \{\xi \in \boxtimes \mid \xi.v = 0 \ \forall v \in V\}$ be the kernel of the \boxtimes -action on V and note that K is an ideal of \boxtimes . By assumption V is nontrivial so $K \neq \boxtimes$. Applying Lemma 4.1 to K we obtain the result. \square

5 Evaluation modules for \boxtimes

In this section we consider a class of finite-dimensional irreducible \boxtimes -modules called *evaluation modules*. We start with a lemma.

Lemma 5.1 *With reference to Definition 2.1 the following (i), (ii) hold for $a \in \mathbb{F} \setminus \{0, 1\}$.*

- (i) *There exists an \mathbb{F} -algebra homomorphism $e_a : \mathcal{A} \rightarrow \mathbb{F}$ that sends $t \mapsto a$.*
- (ii) *There exists a Lie algebra homomorphism $ev_a : \mathfrak{sl}_2 \otimes \mathcal{A} \rightarrow \mathfrak{sl}_2$ that sends $u \otimes b \rightarrow ue_a(b)$ for all $u \in \mathfrak{sl}_2$ and $b \in \mathcal{A}$. Moreover ev_a is surjective.*

Proof: Routine. \square

Definition 5.2 For $a \in \mathbb{F} \setminus \{0, 1\}$ we define $EV_a : \boxtimes \rightarrow \mathfrak{sl}_2$ to be the composition $ev_a \circ \psi$, where ψ and ev_a are from Proposition 2.4 and Lemma 5.1 respectively. We note that EV_a is a surjective homomorphism of Lie algebras.

The map EV_a from Definition 5.2 is characterized as follows.

Lemma 5.3 *For a Lie algebra homomorphism $\varepsilon : \boxtimes \rightarrow \mathfrak{sl}_2$ the following (i), (ii) are equivalent.*

- (i) ε sends

$$x_{12} \mapsto x, \quad x_{23} \mapsto y, \quad x_{31} \mapsto z, \tag{13}$$

where x, y, z is the equitable basis for \mathfrak{sl}_2 .

(ii) There exists $a \in \mathbb{F} \setminus \{0, 1\}$ such that $\varepsilon = EV_a$.

Suppose (i), (ii) hold. Then ε sends

$$x_{03} \mapsto ay + (a - 1)z, \quad (14)$$

$$x_{01} \mapsto (1 - a^{-1})z - a^{-1}x, \quad (15)$$

$$x_{02} \mapsto (1 - a)^{-1}x + a(1 - a)^{-1}y. \quad (16)$$

Proof: (i) \Rightarrow (ii) The map ε is nonzero so Proposition 4.3 applies; let $a \in \mathbb{F}$ be from that proposition and note that $a \neq 0$, $a \neq 1$. We show that $\varepsilon = EV_a$. To this end we first show that ε satisfies (14)–(16). Line (14) follows from Definition 2.2(i), line (13), and the bottom-right display in Proposition 4.3. Line (15) follows from Definition 2.2(i), line (13), and the top-right display in Proposition 4.3. Line (16) follows from Definition 2.2(i), line (13), and the bottom-left display in Proposition 4.3. We have now shown that ε satisfies (14)–(16). Comparing the data in Proposition 2.4 with (13)–(16) we find $\varepsilon(x_{ij}) = EV_a(x_{ij})$ for all distinct $i, j \in \mathbb{I}$. It follows that $\varepsilon = EV_a$.

(ii) \Rightarrow (i) Immediate from Proposition 2.4 and Definition 5.2.

Suppose (i), (ii) hold. We saw in the proof of (i) \Rightarrow (ii) that ε satisfies (14)–(16). \square

Definition 5.4 For a finite-dimensional \mathfrak{sl}_2 -module V and for $a \in \mathbb{F} \setminus \{0, 1\}$ we pull back the \mathfrak{sl}_2 -module action via EV_a to obtain a \boxtimes -module structure on V . We denote this \boxtimes -module by $V(a)$.

We now recall the finite-dimensional irreducible \mathfrak{sl}_2 -modules.

Lemma 5.5 [17, p. 31] *There exists a family*

$$V_d \quad d = 0, 1, 2, \dots \quad (17)$$

of finite-dimensional irreducible \mathfrak{sl}_2 -modules with the following properties. The module V_d has a basis $\{v_n\}_{n=0}^d$ satisfying $h.v_n = (d - 2n)v_n$ for $0 \leq n \leq d$, $f.v_n = (n + 1)v_{n+1}$ for $0 \leq n \leq d - 1$, $f.v_d = 0$, $e.v_n = (d - n + 1)v_{n-1}$ for $1 \leq n \leq d$, $e.v_0 = 0$. Every finite-dimensional irreducible \mathfrak{sl}_2 -module is isomorphic to exactly one of the modules in line (17).

Lemma 5.6 *With respect to the basis $\{v_n\}_{n=0}^d$ for V_d given in Lemma 5.5, the elements x, y, z act as follows. We have $(x + (d - 2n)I).v_n = 2(d - n + 1)v_{n-1}$ for $1 \leq n \leq d$, $(x + dI).v_0 = 0$, $(y + (d - 2n)I).v_n = -2(n + 1)v_{n+1}$ for $0 \leq n \leq d - 1$, $(y - dI).v_d = 0$, $z.v_n = (d - 2n)v_n$ for $0 \leq n \leq d$.*

Proof: Use Lemma 5.5 and (3). \square

Definition 5.7 With reference to Definition 5.4 and Lemma 5.5, by an *evaluation module* for \boxtimes we mean a \boxtimes -module $V_d(a)$ where d is a positive integer and $a \in \mathbb{F} \setminus \{0, 1\}$. We note that the \boxtimes -module $V_d(a)$ is nontrivial and irreducible, with diameter d and shape $(1, 1, \dots, 1)$. We call a the *evaluation parameter* of $V_d(a)$.

Example 5.8 For $a \in \mathbb{F} \setminus \{0, 1\}$ the \boxtimes -module $V_1(a)$ is described as follows. Let v_0, v_1 denote the basis for the \mathfrak{sl}_2 -module V_1 given in Lemma 5.5. With respect to this basis and for the \boxtimes -module $V_1(a)$ the generators x_{ij} are represented by the following matrices.

$$\begin{aligned} x_{12} &: \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, & x_{03} &: \begin{pmatrix} -1 & 0 \\ -2a & 1 \end{pmatrix}, \\ x_{23} &: \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}, & x_{01} &: \begin{pmatrix} 1 & -2a^{-1} \\ 0 & -1 \end{pmatrix}, \\ x_{31} &: \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & x_{02} &: \begin{pmatrix} \frac{a+1}{a-1} & \frac{2}{1-a} \\ \frac{2a}{a-1} & \frac{1+a}{1-a} \end{pmatrix}. \end{aligned}$$

Proof: Combine Lemma 5.3 and Lemma 5.6. □

6 The evaluation modules for \boxtimes ; three characterizations

In this section we give three characterizations of the evaluation modules for \boxtimes . Here is the first characterization.

Proposition 6.1 *Let V denote a nontrivial finite-dimensional irreducible \boxtimes -module. Then for $a \in \mathbb{F} \setminus \{0, 1\}$ the following (i), (ii) are equivalent.*

(i) *V is isomorphic to an evaluation module with evaluation parameter a .*

(ii) *Each of the following vanishes on V :*

$$ax_{01} + (1-a)x_{02} - x_{03}, \quad ax_{10} + (1-a)x_{13} - x_{12}, \quad (18)$$

$$ax_{23} + (1-a)x_{20} - x_{21}, \quad ax_{32} + (1-a)x_{31} - x_{30}. \quad (19)$$

Proof: (i) \Rightarrow (ii) Immediate from Lemma 5.3.

(ii) \Rightarrow (i) By Lemma 2.5 the generators x_{12}, x_{23}, x_{31} form a basis for a subalgebra L of \boxtimes that is isomorphic to \mathfrak{sl}_2 . Let K denote the ideal of \boxtimes generated by the four elements in (18), (19). We claim that K is equal to the kernel of the \boxtimes -action on V . To prove the claim, let K' denote the kernel in question and observe that $K \subseteq K'$. Note that K' is an ideal of \boxtimes ; also $K' \neq \boxtimes$ since V is nontrivial. Applying Lemma 4.1 to K' we obtain $K' \cap L = 0$. Note that $K + L$ is a subalgebra of \boxtimes that contains the generators x_{ij} ($i, j \in \mathbb{I}, i \neq j$). Therefore $K + L = \boxtimes$. By these comments $K = K'$ and the claim is proved. We now show that the \boxtimes -modules V and $V_d(a)$ are isomorphic, where $d+1$ is the dimension of V . For V and $V_d(a)$ we restrict the \boxtimes -action to L . We mentioned earlier that the \boxtimes -module $V_d(a)$ is irreducible. Now the L -module $V_d(a)$ is irreducible since $K + L = \boxtimes$ and K vanishes on $V_d(a)$. Similarly the L -module V is irreducible. Since L is isomorphic to \mathfrak{sl}_2 and since $V, V_d(a)$ have the same dimension the L -modules $V, V_d(a)$ are isomorphic. Let $\mu : V \rightarrow V_d(a)$ denote an isomorphism of L -modules. Since $K + L = \boxtimes$ and K vanishes on each of $V,$

$V_d(a)$ the map μ extends to a \boxtimes -module isomorphism from V to $V_d(a)$. Therefore the \boxtimes -modules V and $V_d(a)$ are isomorphic as desired. \square

We have a comment.

Lemma 6.2 *Two evaluation modules for \boxtimes are isomorphic if and only if they have the same diameter and the same evaluation parameter.*

Proof: Two isomorphic evaluation modules have the same diameter, since they have the same dimension and the dimension is one more than the diameter. Suppose we are given isomorphic evaluation modules $V_d(a)$ and $V_d(b)$. We show $a = b$. By (18) the element $ax_{01} + (1-a)x_{02} - x_{03}$ vanishes on $V_d(a)$. By (18) and since $V_d(a), V_d(b)$ are isomorphic the element $bx_{01} + (1-b)x_{02} - x_{03}$ vanishes on $V_d(a)$. Comparing these elements we find that $a = b$ or $x_{01} - x_{02}$ vanishes on $V_d(a)$. The second possibility contradicts Definition 2.2(i) and Lemma 4.4, so $a = b$ and the result follows. \square

Here is the second characterization of the evaluation modules for \boxtimes .

Proposition 6.3 *Let V denote a nontrivial finite-dimensional irreducible \boxtimes -module. Then the following (i), (ii) are equivalent.*

(i) *V is isomorphic to an evaluation module for \boxtimes .*

(ii) *There exist mutually distinct $i, j, k, \ell \in \mathbb{I}$ such that $x_{ij}, x_{ik}, x_{i\ell}$ are linearly dependent on V .*

Proof: (i) \Rightarrow (ii) Immediate from Proposition 6.1.

(ii) \Rightarrow (i) Without loss we assume $(i, j, k, \ell) = (0, 1, 2, 3)$. By assumption there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$, not all zero, such that

$$\alpha_1 x_{01} + \alpha_2 x_{02} + \alpha_3 x_{03} = 0 \quad (20)$$

on V . Each of $\alpha_1, \alpha_2, \alpha_3$ is nonzero by Definition 2.2(i) and Lemma 4.4. Taking the Lie bracket of x_{01} with (20) and simplifying the result using Definition 2.2(i),(ii) we find that

$$(\alpha_2 + \alpha_3)x_{01} - \alpha_2 x_{02} - \alpha_3 x_{03} = 0 \quad (21)$$

on V . Adding (20), (21) and using Lemma 4.4 we find

$$\alpha_1 + \alpha_2 + \alpha_3 = 0. \quad (22)$$

Taking the Lie bracket of x_{12} with (20) and simplifying the result using Definition 2.2(i),(ii) and (22) we find

$$[x_{12}, x_{03}] = 2\alpha_1 \alpha_3^{-1} x_{01} - 2\alpha_2 \alpha_3^{-1} x_{02} - 2x_{12} \quad (23)$$

on V . Cyclically permuting 1, 2, 3 in the previous argument we find that both

$$[x_{23}, x_{01}] = 2\alpha_2 \alpha_1^{-1} x_{02} - 2\alpha_3 \alpha_1^{-1} x_{03} - 2x_{23}, \quad (24)$$

$$[x_{31}, x_{02}] = 2\alpha_3 \alpha_2^{-1} x_{03} - 2\alpha_1 \alpha_2^{-1} x_{01} - 2x_{31} \quad (25)$$

on V . By the Jacobi identity

$$[x_{01}, [x_{12}, x_{23}]] + [x_{12}, [x_{23}, x_{01}]] + [x_{23}, [x_{01}, x_{12}]] = 0. \quad (26)$$

Evaluating (26) using Definition 2.2(i),(ii) and (22)–(24) we find that

$$\alpha_1 x_{23} + \alpha_2 x_{20} + \alpha_3 x_{21} = 0 \quad (27)$$

on V . Cyclically permuting 1, 2, 3 in the previous argument we find that both

$$\alpha_2 x_{31} + \alpha_3 x_{30} + \alpha_1 x_{32} = 0, \quad (28)$$

$$\alpha_3 x_{12} + \alpha_1 x_{10} + \alpha_2 x_{13} = 0 \quad (29)$$

on V . By (20), (22) and (27)–(29) the four expressions (18), (19) vanish on V , where $a = -\alpha_1 \alpha_3^{-1}$. Note that $a \neq 0$ since $\alpha_1 \neq 0$, and $a \neq 1$ since $\alpha_2 \neq 0$. Now by Proposition 6.1, V is isomorphic to an evaluation module for \boxtimes with evaluation parameter a . The result follows. \square

Here is the third characterization of the evaluation modules for \boxtimes .

Proposition 6.4 *Let V denote a nontrivial finite-dimensional irreducible \boxtimes -module. Then the following (i), (ii) are equivalent.*

(i) V is isomorphic to an evaluation module for \boxtimes .

(ii) V has shape $(1, 1, \dots, 1)$.

Proof: (i) \Rightarrow (ii) By Definition 5.7 each evaluation module for \boxtimes has shape $(1, 1, \dots, 1)$.
(ii) \Rightarrow (i) Throughout this proof the decomposition $[1, 3]$ of V will be denoted by $\{V_n\}_{n=0}^d$. By definition, for $0 \leq n \leq d$ the space V_n is an eigenspace for x_{13} with eigenvalue $2n - d$. By this and since $x_{13} + x_{31} = 0$, the space V_n is an eigenspace for x_{31} with eigenvalue $d - 2n$. Also V_n has dimension 1 by our assumption on the shape. We will need the action of x_{30} on $\{V_n\}_{n=0}^d$. By the table in Section 3,

$$(x_{30} - (d - 2n)I)V_n \subseteq V_{n+1} \quad (0 \leq n \leq d), \quad (30)$$

where we recall $V_{d+1} = 0$. By Lemma 2.5 there exists an injective homomorphism of Lie algebras from \mathfrak{sl}_2 to \boxtimes that sends x, y, z to x_{12}, x_{23}, x_{31} respectively. Using this homomorphism we pull back the \boxtimes -module structure on V to obtain an \mathfrak{sl}_2 -module structure on V . Since z and x_{31} agree on V we find that for $0 \leq n \leq d$, V_n is an eigenspace for z with eigenvalue $d - 2n$. These eigenspaces all have dimension 1 so the \mathfrak{sl}_2 -module V is irreducible. Therefore the \mathfrak{sl}_2 -module V is isomorphic to the \mathfrak{sl}_2 -module V_d from Lemma 5.5. Let $\{v_n\}_{n=0}^d$ denote the basis for V given in Lemma 5.5. By Lemma 5.6 we find that for $0 \leq n \leq d$ the vector v_n is an eigenvector for z with eigenvalue $d - 2n$; therefore v_n is a basis for V_n . By Lemma 5.6 and since y, x_{23} agree on V , we find $(x_{23} + (d - 2n)I).v_n = -2(n + 1)v_{n+1}$ for $0 \leq n \leq d - 1$ and $(x_{23} - dI).v_d = 0$. By (30) there exist scalars $\{\alpha_n\}_{n=1}^d$ in \mathbb{F} such that

$(x_{30} - (d - 2n)I).v_n = \alpha_{n+1}v_{n+1}$ for $0 \leq n \leq d - 1$ and $(x_{30} + dI).v_d = 0$. By Definition 2.2(ii),

$$[x_{23}, x_{30}] = 2x_{23} + 2x_{30}.$$

For $0 \leq n \leq d$ we apply each side of this equation to v_n and evaluate the result using the above comments. The resulting equations show that α_n/n is independent of n for $1 \leq n \leq d$. Denoting this common value by $2a$ we find $(x_{30} - (d - 2n)I).v_n = 2a(n + 1)v_{n+1}$ for $0 \leq n \leq d - 1$. From our comments so far and since $x_{23} + x_{32} = 0$, the element

$$ax_{32} + (1 - a)x_{31} - x_{30}$$

vanishes on each of $\{v_n\}_{n=0}^d$ and hence on V . In particular the actions of x_{30} , x_{31} , x_{32} on V are linearly dependent. Invoking Proposition 6.3 we find V is isomorphic to an evaluation module for \boxtimes . \square

7 Some automorphisms of \boxtimes

In this section we consider how the symmetric group S_4 acts on \boxtimes as a group of automorphisms. We investigate what happens when we twist an evaluation module for \boxtimes via an element of S_4 .

For the rest of this paper we identify S_4 with the group of permutations of \mathbb{I} . We use the cycle notation; for example $(1, 2, 3)$ denotes the element of S_4 that sends $1 \mapsto 2 \mapsto 3 \mapsto 1$ and $0 \mapsto 0$. Note that S_4 acts on the set of generators for \boxtimes by permuting the indices:

$$\sigma(x_{ij}) = x_{\sigma(i), \sigma(j)} \quad \sigma \in S_4, \quad i, j \in \mathbb{I}, \quad i \neq j.$$

This action leaves invariant the defining relations for \boxtimes and therefore induces an action of S_4 on \boxtimes as a group of automorphisms. This S_4 -action on \boxtimes induces an action of S_4 on the set of \boxtimes -modules, as we now explain.

Definition 7.1 Let V denote a \boxtimes -module. For $\sigma \in S_4$ there exists a \boxtimes -module structure on V , called V *twisted via* σ , that behaves as follows: for all $\xi \in \boxtimes$ and $v \in V$, the vector $\xi.v$ computed in V twisted via σ coincides with the vector $\sigma^{-1}(\xi).v$ computed in the original \boxtimes -module V . Sometimes we abbreviate ${}^\sigma V$ for V twisted via σ . Observe that S_4 acts on the set of \boxtimes -modules, with σ sending V to ${}^\sigma V$ for all $\sigma \in S_4$ and all \boxtimes -modules V .

For the following three lemmas the proofs are routine and left to the reader.

Lemma 7.2 *Let V denote a finite-dimensional irreducible \boxtimes -module. For distinct $i, j \in \mathbb{I}$ and for $\sigma \in S_4$ the following coincide:*

- (i) *The decomposition $[i, j]$ of V ;*
- (ii) *The decomposition $[\sigma(i), \sigma(j)]$ of V twisted via σ .*

Lemma 7.3 *Let V denote a finite-dimensional irreducible \boxtimes -module. For $i \in \mathbb{I}$ and for $\sigma \in S_4$ the following coincide:*

- (i) *The flag $[i]$ of V ;*
- (ii) *The flag $[\sigma(i)]$ of V twisted via σ .*

Lemma 7.4 *Let V denote a finite-dimensional irreducible \boxtimes -module. For $\sigma \in S_4$ the following coincide:*

- (i) *The shape of V ;*
- (ii) *The shape of V twisted via σ .*

Corollary 7.5 *Let V denote an evaluation module for \boxtimes , and pick $\sigma \in S_4$. Then V twisted via σ is isomorphic to an evaluation module for \boxtimes .*

Proof: Combine Proposition 6.4 and Lemma 7.4. □

For each integer $d \geq 1$ let Δ_d denote the set of isomorphism classes of evaluation modules for \boxtimes that have diameter d . By Definition 5.7 and Lemma 6.2 the map $a \mapsto V_d(a)$ induces a bijection from $\mathbb{F} \setminus \{0, 1\}$ to Δ_d . Via this bijection the S_4 -action on Δ_d from Corollary 7.5 induces an S_4 -action on $\mathbb{F} \setminus \{0, 1\}$. This action is described in the following lemma and subsequent theorem.

Lemma 7.6 *There exists an S_4 -action on the set $\mathbb{F} \setminus \{0, 1\}$ that does the following. For all $a \in \mathbb{F} \setminus \{0, 1\}$,*

- (i) $(2, 0)$ sends $a \mapsto a^{-1}$;
- (ii) $(0, 1)$ sends $a \mapsto a(a - 1)^{-1}$;
- (iii) $(1, 3)$ sends $a \mapsto a^{-1}$.

Proof: Consider the following maps on $\mathbb{F} \setminus \{0, 1\}$:

$$\sigma_1 : a \mapsto a^{-1}, \quad \sigma_2 : a \mapsto a(a - 1)^{-1}, \quad \sigma_3 : a \mapsto a^{-1}.$$

One routinely verifies that $\sigma_1^2 = 1$, $\sigma_2^2 = 1$, $\sigma_3^2 = 1$ and that $(\sigma_1\sigma_2)^3 = 1$, $(\sigma_2\sigma_3)^3 = 1$, $(\sigma_1\sigma_3)^2 = 1$. The above equations are the defining relations for a well-known presentation for S_4 [18, p. 105] and the result follows. □

Theorem 7.7 *For an integer $d \geq 1$, for $\sigma \in S_4$, and for $a \in \mathbb{F} \setminus \{0, 1\}$ the following are isomorphic:*

- (i) *the \boxtimes -module $V_d(a)$ twisted via σ ;*
- (ii) *the \boxtimes -module $V_d(\sigma(a))$.*

Proof: We abbreviate $W = V_d(a)$. Without loss we assume that σ is one of $(2, 0)$, $(0, 1)$, $(1, 3)$. In each case we verify using Proposition 6.1 that each of the following vanishes on W :

$$\begin{aligned} &\sigma(a)x_{\sigma(0),\sigma(1)} + (1 - \sigma(a))x_{\sigma(0),\sigma(2)} - x_{\sigma(0),\sigma(3)}, \\ &\sigma(a)x_{\sigma(1),\sigma(0)} + (1 - \sigma(a))x_{\sigma(1),\sigma(3)} - x_{\sigma(1),\sigma(2)}, \\ &\sigma(a)x_{\sigma(2),\sigma(3)} + (1 - \sigma(a))x_{\sigma(2),\sigma(0)} - x_{\sigma(2),\sigma(1)}, \\ &\sigma(a)x_{\sigma(3),\sigma(2)} + (1 - \sigma(a))x_{\sigma(3),\sigma(1)} - x_{\sigma(3),\sigma(0)}. \end{aligned}$$

Now by Definition 7.1 and $\sigma^2 = 1$, each of the following vanishes on ${}^\sigma W$:

$$\begin{aligned} \sigma(a)x_{01} + (1 - \sigma(a))x_{02} - x_{03}, & \quad \sigma(a)x_{10} + (1 - \sigma(a))x_{13} - x_{12}, \\ \sigma(a)x_{23} + (1 - \sigma(a))x_{20} - x_{21}, & \quad \sigma(a)x_{32} + (1 - \sigma(a))x_{31} - x_{30}. \end{aligned}$$

Now by Proposition 6.1 the \boxtimes -module ${}^\sigma W$ has evaluation parameter $\sigma(a)$ and is therefore isomorphic to $V_d(\sigma(a))$. \square

8 The group G

In Lemma 7.6 we gave an action of S_4 on $\mathbb{F} \setminus \{0, 1\}$. In this section we find the kernel of this action. We will use the following notation.

Definition 8.1 A partition of \mathbb{I} into two disjoint sets, each with two elements, is said to have shape $(2, 2)$. Let P denote the set of partitions of \mathbb{I} that have shape $(2, 2)$, and note that P has cardinality 3. The action of S_4 on \mathbb{I} induces an action of S_4 on P and this yields a surjective homomorphism of groups $S_4 \rightarrow S_3$. Let G denote the kernel of this homomorphism. We note that G has cardinality 4 and consists of

$$(01)(23), \quad (02)(13), \quad (03)(12)$$

together with the identity element. Observe that G is isomorphic to the Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Lemma 8.2 *With reference to Definition 7.6, for $\sigma \in S_4$ the following (i), (ii) are equivalent:*

- (i) $\sigma \in G$;
- (ii) $\sigma(a) = a$ for all $a \in \mathbb{F} \setminus \{0, 1\}$.

Proof: Let G' denote the set of elements in S_4 that fix each element of $\mathbb{F} \setminus \{0, 1\}$. We show $G' = G$. The set $\{3, 1/3, -2, -1/2, 3/2, 2/3\}$ is an orbit of S_4 ; therefore the index $[S_4 : G'] \geq 6$ so G' has at most 4 elements. Also G' contains $(2, 0)(1, 3)$ by Lemma 7.6 and G' is normal in S_4 so $G \subseteq G'$. It follows that $G' = G$. \square

Corollary 8.3 *For an integer $d \geq 1$, for $\sigma \in G$, and for $a \in \mathbb{F} \setminus \{0, 1\}$ the following are isomorphic:*

- (i) *the \boxtimes -module $V_d(a)$ twisted via σ ;*
- (ii) *the \boxtimes -module $V_d(a)$.*

Proof: Combine Theorem 7.7 and Lemma 8.2. □

9 The relatives of an evaluation parameter

In Lemma 7.6 we considered an S_4 -action on the set $\mathbb{F} \setminus \{0, 1\}$. In this section we describe the orbits of this action. We start with a definition.

Definition 9.1 Pick $a \in \mathbb{F} \setminus \{0, 1\}$ and mutually distinct $i, j, k, \ell \in \mathbb{I}$. By the (i, j, k, ℓ) -relative of a we mean the scalar $\sigma(a)$ where $\sigma \in S_4$ satisfies $\sigma(i) = 2$, $\sigma(j) = 0$, $\sigma(k) = 1$, $\sigma(\ell) = 3$.

Lemma 9.2 *Pick $a \in \mathbb{F} \setminus \{0, 1\}$ and mutually distinct $i, j, k, \ell \in \mathbb{I}$. Let α denote the (i, j, k, ℓ) -relative of a . Then (i)–(iii) hold below.*

- (i) α^{-1} is the (j, i, k, ℓ) -relative of a ;
- (ii) $\alpha(\alpha - 1)^{-1}$ is the (i, k, j, ℓ) -relative of a ;
- (iii) α^{-1} is the (i, j, ℓ, k) -relative of a .

Proof: By Definition 9.1 there exists $\sigma \in S_4$ that sends the sequence $(a; i, j, k, \ell)$ to $(\alpha; 2, 0, 1, 3)$. To obtain (i), note that $(2, 0)\sigma$ sends $(a; j, i, k, \ell)$ to $(\alpha^{-1}; 2, 0, 1, 3)$. To obtain (ii), note that $(0, 1)\sigma$ sends $(a; i, k, j, \ell)$ to $(\alpha(\alpha - 1)^{-1}; 2, 0, 1, 3)$. To obtain (iii), note that $(1, 3)\sigma$ sends $(a; i, j, \ell, k)$ to $(\alpha^{-1}; 2, 0, 1, 3)$. □

In the following lemma we interpret each relative as a cross ratio (see [7, p. 48] for the details).

Lemma 9.3 *For $a \in \mathbb{F} \setminus \{0, 1\}$ and mutually distinct $i, j, k, \ell \in \mathbb{I}$ the following (i), (ii) coincide:*

- (i) *the (i, j, k, ℓ) -relative of a ;*
- (ii) *the scalar $\frac{\hat{i}-\hat{\ell}}{\hat{i}-\hat{k}} \frac{\hat{j}-\hat{k}}{\hat{j}-\hat{\ell}}$ where we define $\hat{0} = a$, $\hat{1} = 0$, $\hat{2} = 1$, $\hat{3} = \infty$.*

Proof: Let us call the scalar in (ii) the (i, j, k, ℓ) -partner of a . Denoting this partner by α we observe:

- α^{-1} is the (j, i, k, ℓ) -partner of a ;

- $\alpha(\alpha - 1)^{-1}$ is the (i, k, j, ℓ) -partner of a ;
- α^{-1} is the (i, j, ℓ, k) -partner of a .

Note that a is the $(2, 0, 1, 3)$ -partner of a as well as the $(2, 0, 1, 3)$ -relative of a . By these comments and Lemma 9.2 we find that the partner function and the relative function satisfy the same recursion and same initial condition. Therefore these functions coincide and the result follows. \square

Proposition 9.4 *Pick $a \in \mathbb{F} \setminus \{0, 1\}$ and mutually distinct $i, j, k, \ell \in \mathbb{I}$. Then the (i, j, k, ℓ) -relative of a is given in the following table.*

| (i, j, k, ℓ) | | | | (i, j, k, ℓ) -relative |
|-------------------|----------------|----------------|----------------|-----------------------------|
| $(2, 0, 1, 3)$ | $(0, 2, 3, 1)$ | $(1, 3, 2, 0)$ | $(3, 1, 0, 2)$ | a |
| $(0, 2, 1, 3)$ | $(2, 0, 3, 1)$ | $(1, 3, 0, 2)$ | $(3, 1, 2, 0)$ | a^{-1} |
| $(1, 0, 2, 3)$ | $(0, 1, 3, 2)$ | $(2, 3, 1, 0)$ | $(3, 2, 0, 1)$ | $1 - a$ |
| $(0, 1, 2, 3)$ | $(1, 0, 3, 2)$ | $(2, 3, 0, 1)$ | $(3, 2, 1, 0)$ | $(1 - a)^{-1}$ |
| $(2, 1, 0, 3)$ | $(1, 2, 3, 0)$ | $(0, 3, 2, 1)$ | $(3, 0, 1, 2)$ | $a(a - 1)^{-1}$ |
| $(1, 2, 0, 3)$ | $(2, 1, 3, 0)$ | $(0, 3, 1, 2)$ | $(3, 0, 2, 1)$ | $1 - a^{-1}$ |

Proof: Routine calculation using Lemma 9.2 or Lemma 9.3. \square

10 24 bases for an evaluation module

Let V denote an evaluation module for \boxtimes . In this section we display 24 bases for V that we find attractive. We compute the action of the generators x_{ij} on each of these bases. We start with a definition.

Definition 10.1 Let V denote an evaluation module for \boxtimes and pick mutually distinct $i, j, k, \ell \in \mathbb{I}$. A basis $\{u_n\}_{n=0}^d$ of V is called an $[i, j, k, \ell]$ -basis whenever (i), (ii) hold below:

- (i) for $0 \leq n \leq d$ the vector u_n is contained in component n of the decomposition $[k, \ell]$ for V ;
- (ii) $\sum_{n=0}^d u_n$ is contained in component 0 of the flag $[i]$ for V .

Referring to Definition 10.1, we will show that V has an $[i, j, k, \ell]$ -basis after a few comments.

Lemma 10.2 *Let V denote an evaluation module for \boxtimes and pick mutually distinct $i, j, k, \ell \in \mathbb{I}$. Let $\{u_n\}_{n=0}^d$ denote an $[i, j, k, \ell]$ -basis for V , and for $0 \leq n \leq d$ let u'_n denote any vector in V . Then the following are equivalent:*

- (i) the sequence $\{u'_n\}_{n=0}^d$ is an $[i, j, k, \ell]$ -basis for V ;

(ii) there exists a nonzero $\beta \in \mathbb{F}$ such that $u'_n = \beta u_n$ for $0 \leq n \leq d$.

Proof: (i) \Rightarrow (ii) Recall that V has shape $(1, 1, \dots, 1)$.

(ii) \Rightarrow (i) Immediate from Definition 10.1. \square

Lemma 10.3 Let $V_d(a)$ denote an evaluation module for \boxtimes . Then the basis $\{v_n\}_{n=0}^d$ for the \mathfrak{sl}_2 -module V_d from Definition 5.5 is a $[2, 0, 1, 3]$ -basis for $V_d(a)$.

Proof: By (13) and since $x_{13} = -x_{31}$ we find $z + x_{13}$ vanishes on $V_d(a)$. Now by Lemma 5.6, for $0 \leq n \leq d$ we have $x_{13}.v_n = (2n - d)v_n$ so v_n is contained in component n of the decomposition $[1, 3]$ for $V_d(a)$. Define $\eta = \sum_{n=0}^d v_n$. Using (13) and Lemma 5.6 we find $x_{23}.\eta = -d\eta$; therefore η is contained in component 0 of the flag $[2]$ for $V_d(a)$. Now the basis $\{v_n\}_{n=0}^d$ is a $[2, 0, 1, 3]$ -basis by Definition 10.1. \square

Lemma 10.4 Let V denote an evaluation module for \boxtimes and pick mutually distinct $i, j, k, \ell \in \mathbb{I}$. Then for $\sigma \in S_4$ the following are the same:

(i) an $[i, j, k, \ell]$ -basis for V ;

(ii) a $[\sigma(i), \sigma(j), \sigma(k), \sigma(\ell)]$ -basis for V twisted via σ .

Proof: Use Lemma 7.2 and Lemma 7.3. \square

Proposition 10.5 Let V denote an evaluation module for \boxtimes and pick mutually distinct $i, j, k, \ell \in \mathbb{I}$. Then there exists an $[i, j, k, \ell]$ -basis for V .

Proof: Let σ denote the element of S_4 that sends the sequence (i, j, k, ℓ) to $(2, 0, 1, 3)$. The \boxtimes -module ${}^\sigma V$ is isomorphic to an evaluation module by Corollary 7.5, so ${}^\sigma V$ has a $[2, 0, 1, 3]$ -basis by Lemma 10.3. Now V has an $[i, j, k, \ell]$ -basis in view of Lemma 10.4. \square

We now find the action of each generator x_{rs} on the bases given in Proposition 10.5. To describe these actions we use the following notation. For an integer $d \geq 0$ let $\text{Mat}_{d+1}(\mathbb{F})$ denote the \mathbb{F} -algebra of all $d + 1$ by $d + 1$ matrices that have entries in \mathbb{F} . We index the rows and columns by $0, 1, \dots, d$. Let V denote a vector space over \mathbb{F} with dimension $d + 1$ and let $\{u_n\}_{n=0}^d$ denote a basis for V . For a linear transformation $A : V \rightarrow V$ there exists a unique $X \in \text{Mat}_{d+1}(\mathbb{F})$ such that $A.u_j = \sum_{i=0}^d X_{ij}u_i$ for $0 \leq j \leq d$. We call X the *matrix that represents A with respect to $\{u_n\}_{n=0}^d$* .

Lemma 10.6 Let $V_d(a)$ denote an evaluation module for \boxtimes . For mutually distinct $i, j, k, \ell \in \mathbb{I}$, for distinct $r, s \in \mathbb{I}$, and for $\sigma \in S_4$ the following are the same:

(i) the matrix that represents x_{rs} with respect to an $[i, j, k, \ell]$ -basis for $V_d(a)$;

(ii) the matrix that represents $x_{\sigma(r), \sigma(s)}$ with respect to a $[\sigma(i), \sigma(j), \sigma(k), \sigma(\ell)]$ -basis for $V_d(\sigma(a))$.

Proof: Routine using Definition 7.1, Theorem 7.7, and Lemma 10.4. \square

Theorem 10.7 *Let $V_d(a)$ denote an evaluation module for \boxtimes . For mutually distinct $i, j, k, \ell \in \mathbb{I}$ and distinct $r, s \in \mathbb{I}$ consider the matrix that represents x_{rs} with respect to an $[i, j, k, \ell]$ -basis for $V_d(a)$. The entries of this matrix are given in the following table. All entries not displayed are zero.*

| generator | $(n, n-1)$ -entry | (n, n) -entry | $(n-1, n)$ -entry |
|--------------|----------------------------|-----------------------------------|---------------------------|
| $x_{\ell k}$ | 0 | $d-2n$ | 0 |
| $x_{k\ell}$ | 0 | $2n-d$ | 0 |
| x_{ki} | 0 | $2n-d$ | $2d-2n+2$ |
| x_{ik} | 0 | $d-2n$ | $2n-2d-2$ |
| $x_{i\ell}$ | $-2n$ | $2n-d$ | 0 |
| $x_{\ell i}$ | $2n$ | $d-2n$ | 0 |
| $x_{\ell j}$ | $2\alpha n$ | $d-2n$ | 0 |
| $x_{j\ell}$ | $-2\alpha n$ | $2n-d$ | 0 |
| x_{jk} | 0 | $d-2n$ | $2(n-d-1)\alpha^{-1}$ |
| x_{kj} | 0 | $2n-d$ | $2(d-n+1)\alpha^{-1}$ |
| x_{ji} | $2\alpha n(\alpha-1)^{-1}$ | $(d-2n)(\alpha+1)(\alpha-1)^{-1}$ | $2(d-n+1)(1-\alpha)^{-1}$ |
| x_{ij} | $2\alpha n(1-\alpha)^{-1}$ | $(d-2n)(\alpha+1)(1-\alpha)^{-1}$ | $2(d-n+1)(\alpha-1)^{-1}$ |

In the above table the scalar α denotes the (i, j, k, ℓ) -relative of a from Definition 9.1.

Proof: First consider the special case $(i, j, k, \ell) = (2, 0, 1, 3)$. Without loss we take our $[2, 0, 1, 3]$ -basis to be the basis $\{v_n\}_{n=0}^d$ from Lemma 10.3. We then obtain the action of x_{rs} on $\{v_n\}_{n=0}^d$ using Lemma 5.3 and Lemma 5.6. This gives the result for the special case. To get the result for general (i, j, k, ℓ) , let σ denote the element of S_4 that sends the sequence $(a; i, j, k, \ell)$ to $(\alpha; 2, 0, 1, 3)$. Using σ and Lemma 10.6 we reduce to the above special case. \square

11 A bilinear form for the evaluation modules

Let V denote an evaluation module for \boxtimes and consider the 24 bases for V from Definition 10.1. In Section 13 we will compute the transition matrices between certain pairs of bases among these 24. Before we get to this, it is convenient to introduce a bilinear form on V .

We recall a few concepts from linear algebra. Let V denote a vector space over \mathbb{F} with finite positive dimension. By a *bilinear form* on V we mean a map $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$ that satisfies the following four conditions for all $u, v, w \in V$ and for all $\alpha \in \mathbb{F}$: (i) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$; (ii) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$; (iii) $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$; (iv) $\langle u, \alpha v \rangle = \alpha \langle u, v \rangle$. We observe that a scalar multiple of a bilinear form on V is a bilinear form on V . Let $\langle \cdot, \cdot \rangle$ denote a bilinear form on V . This form is said to be *symmetric* (resp. *antisymmetric*) whenever $\langle u, v \rangle = \langle v, u \rangle$ (resp. $\langle u, v \rangle = -\langle v, u \rangle$) for all $u, v \in V$. Let $\langle \cdot, \cdot \rangle$ denote a bilinear form on V .

Then the following are equivalent: (i) there exists a nonzero $u \in V$ such that $\langle u, v \rangle = 0$ for all $v \in V$; (ii) there exists a nonzero $v \in V$ such that $\langle u, v \rangle = 0$ for all $u \in V$. The form \langle, \rangle is said to be *degenerate* whenever (i), (ii) hold and *nondegenerate* otherwise. Assume \langle, \rangle is nondegenerate. Let $\{U_n\}_{n=0}^d$ and $\{V_n\}_{n=0}^d$ denote decompositions of V that have the same diameter. These decompositions are called *dual* with respect to \langle, \rangle whenever $\langle U_i, V_j \rangle = 0$ for $0 \leq i, j \leq d, i \neq j$. In this case $\{U_n\}_{n=0}^d$ and $\{V_n\}_{n=0}^d$ have the same shape.

Lemma 11.1 *For each integer $d \geq 0$ there exists a nonzero bilinear form \langle, \rangle on the \mathfrak{sl}_2 -module V_d such that*

$$\langle \xi.u, v \rangle = -\langle u, \xi.v \rangle \quad \xi \in \mathfrak{sl}_2, \quad u, v \in V_d.$$

The form is nondegenerate. The form is unique up to multiplication by a nonzero scalar in \mathbb{F} . The form is symmetric (resp. antisymmetric) when d is even (resp. odd).

Proof: Concerning existence, let $\{v_n\}_{n=0}^d$ denote the basis for V_d from Lemma 5.5, and let \langle, \rangle denote the bilinear form on V_d that satisfies

$$\langle v_r, v_s \rangle = \delta_{r+s,d} (-1)^r \binom{d}{r} \quad (0 \leq r, s \leq d).$$

Using the data in Lemma 5.5 we find $\langle \xi.v_r, v_s \rangle = -\langle v_r, \xi.v_s \rangle$ for $\xi \in \{e, f, h\}$ and $0 \leq r, s \leq d$. Since $\{v_n\}_{n=0}^d$ is a basis for V_d and e, f, h is a basis for \mathfrak{sl}_2 we conclude $\langle \xi.u, v \rangle = -\langle u, \xi.v \rangle$ for $\xi \in \mathfrak{sl}_2$ and $u, v \in V_d$. This shows that the required bilinear form exists. The remaining assertions are routinely verified. \square

Lemma 11.2 *Let V denote an evaluation module for \boxtimes . Then there exists a nonzero bilinear form \langle, \rangle on V such that*

$$\langle \xi.u, v \rangle = -\langle u, \xi.v \rangle \quad \xi \in \boxtimes, \quad u, v \in V.$$

The form is nondegenerate. The form is unique up to multiplication by a nonzero scalar in \mathbb{F} . The form is symmetric (resp. antisymmetric) when the diameter of V is even (resp. odd).

Proof: By definition there exist an integer $d \geq 1$ and $a \in \mathbb{F} \setminus \{0, 1\}$ such that $V = V_d(a)$. Let \langle, \rangle denote the bilinear form on the \mathfrak{sl}_2 -module V_d from Lemma 11.1. This form meets the requirements of the present lemma since each element of \boxtimes acts on $V_d(a)$ as an element of \mathfrak{sl}_2 . This shows that the required bilinear form exists. The remaining assertions follow from Lemma 11.1 and since $EV_a : \boxtimes \rightarrow \mathfrak{sl}_2$ is surjective. \square

Definition 11.3 Referring to Lemma 11.2 we call \langle, \rangle a *standard bilinear form* for V .

We have a comment.

Lemma 11.4 *Let V denote an evaluation module for \boxtimes . Then for $\sigma \in S_4$ the following (i), (ii) are the same:*

- (i) *a standard bilinear form for V ;*
- (ii) *a standard bilinear form for V twisted via σ .*

Proof: Combine Definition 7.1 and Lemma 11.2. □

Lemma 11.5 *Let V denote an evaluation module for \boxtimes with standard bilinear form $\langle \cdot, \cdot \rangle$. Then for distinct $i, j \in \mathbb{I}$ the decompositions $[i, j]$ and $[j, i]$ of V are dual with respect to $\langle \cdot, \cdot \rangle$.*

Proof: Let $\{V_n\}_{n=0}^d$ denote the decomposition $[i, j]$ and note that $\{V_{d-n}\}_{n=0}^d$ is the decomposition $[j, i]$. For $0 \leq r, s \leq d$ with $r + s \neq d$ we show V_r, V_s are orthogonal with respect to $\langle \cdot, \cdot \rangle$. For $u \in V_r$ and $v \in V_s$ we have $\langle x_{ij}.u, v \rangle = -\langle u, x_{ij}.v \rangle$; evaluating this using $x_{ij}.u = (2r - d)u$ and $x_{ij}.v = (2s - d)v$ we obtain $(r + s - d)\langle u, v \rangle = 0$. We assume $r + s \neq d$ so $\langle u, v \rangle = 0$ and therefore V_r, V_s are orthogonal with respect to $\langle \cdot, \cdot \rangle$. The result follows. □

Lemma 11.6 *Let $V_d(a)$ denote an evaluation module for \boxtimes with standard bilinear form $\langle \cdot, \cdot \rangle$. Then for $i \in \mathbb{I}$ and for $0 \leq n \leq d - 1$ the following (i), (ii) are orthogonal complements with respect to $\langle \cdot, \cdot \rangle$:*

- (i) *component n of the flag $[i]$;*
- (ii) *component $d - n - 1$ of the flag $[i]$.*

Proof: Fix $j \in \mathbb{I}$ ($j \neq i$) and let $\{V_n\}_{n=0}^d$ denote the decomposition $[i, j]$ of $V_d(a)$. By construction, component n (resp. component $d - n - 1$) of the flag $[i]$ is $V_0 + V_1 + \cdots + V_n$ (resp. $V_0 + V_1 + \cdots + V_{d-n-1}$). These components are orthogonal by Lemma 11.5. Moreover the sum of their dimensions is $d + 1$ and this equals the dimension of $V_d(a)$. Therefore these components are orthogonal complements. □

Proposition 11.7 *Let V denote an evaluation module for \boxtimes with standard bilinear form $\langle \cdot, \cdot \rangle$. Pick mutually distinct $i, j, k, \ell \in \mathbb{I}$ and consider an $[i, j, k, \ell]$ -basis for V from Definition 10.1. Denoting this basis by $\{u_n\}_{n=0}^d$ we have*

$$\langle u_r, u_s \rangle = \delta_{r+s, d} (-1)^r \binom{d}{r} \langle u_0, u_d \rangle \quad (31)$$

for $0 \leq r, s \leq d$.

Proof: If $r + s \neq d$ then $\langle u_r, u_s \rangle = 0$ by Lemma 11.5. Also by Lemma 11.2 we have

$$\langle x_{ki}.u_n, u_{d-n+1} \rangle = -\langle u_n, x_{ki}.u_{d-n+1} \rangle \quad (32)$$

for $1 \leq n \leq d$. The action of x_{ki} on $\{u_n\}_{n=0}^d$ is given in Theorem 10.7. Evaluating (32) using this data we find

$$(d - n + 1)\langle u_{n-1}, u_{d-n+1} \rangle = -n\langle u_n, u_{d-n} \rangle \quad (1 \leq n \leq d).$$

Solving this recursion we find

$$\langle u_r, u_{d-r} \rangle = (-1)^r \binom{d}{r} \langle u_0, u_d \rangle \quad (0 \leq r \leq d).$$

The result follows. \square

12 A normalization for the 24 bases

Let V denote an evaluation module for \boxtimes and consider the 24 bases for V from Definition 10.1. In the next section we will compute the transition matrices between certain pairs of bases among these 24. In order to do this elegantly we first normalize our bases.

Notation 12.1 We fix an integer $d \geq 1$ and a scalar $a \in \mathbb{F} \setminus \{0, 1\}$. We consider the \boxtimes -module $V = V_d(a)$ from Definition 5.7. For $i \in \mathbb{I}$ we fix a nonzero vector η_i in component 0 of the flag $[i]$ for V . Let \langle, \rangle denote a standard bilinear form on V .

Lemma 12.2 *With reference to Notation 12.1, for mutually distinct $i, j, k, \ell \in \mathbb{I}$ there exists a unique basis $\{u_n\}_{n=0}^d$ of V such that (i), (ii) hold below:*

(i) *for $0 \leq n \leq d$ the vector u_n is contained in component n of the decomposition $[k, \ell]$ for V ;*

(ii) $\eta_i = \sum_{n=0}^d u_n$.

We denote this basis by $[i, j, k, \ell]$.

Proof: By Proposition 10.5 there exists an $[i, j, k, \ell]$ -basis for V . Denote this basis by $\{u'_n\}_{n=0}^d$. By Definition 10.1, for $0 \leq n \leq d$ the vector u'_n is in component n of the decomposition $[k, \ell]$. Also $\sum_{n=0}^d u'_n$ lies in component 0 of the flag $[i]$. Define $\eta = \sum_{n=0}^d u'_n$ and note that $\eta \neq 0$. Each of η, η_i span component 0 of the flag $[i]$ so there exists a nonzero $\beta \in \mathbb{F}$ such that $\eta = \beta\eta_i$. Define $u_n = \beta^{-1}u'_n$ for $0 \leq n \leq d$. By construction $\{u_n\}_{n=0}^d$ is a basis for V that satisfies (i), (ii) above. This shows that the required basis exists. This basis is unique by condition (ii) above and Lemma 10.2. \square

Lemma 12.3 *With reference to Notation 12.1, the following (i), (ii) hold.*

(i) $\langle \eta_i, \eta_i \rangle = 0$ for $i \in \mathbb{I}$;

(ii) $\langle \eta_i, \eta_j \rangle \neq 0$ for distinct $i, j \in \mathbb{I}$.

Proof: (i) The vector η_i is in component 0 of the flag $[i]$ on V . This component is orthogonal to itself by Lemma 11.6 and since $d \geq 1$.

(ii) Let $\{V_n\}_{n=0}^d$ denote the decomposition $[i, j]$ of V . By construction η_i spans V_0 and η_j spans V_d . By Lemma 11.6 the orthogonal complement of V_0 is $V_0 + \cdots + V_{d-1}$. Therefore V_0, V_d are not orthogonal and this implies $\langle \eta_i, \eta_j \rangle \neq 0$. \square

Lemma 12.4 *With reference to Notation 12.1, pick mutually distinct $i, j, k, \ell \in \mathbb{I}$ and consider the basis $[i, j, k, \ell]$ for V from Lemma 12.2. Denoting this basis by $\{u_n\}_{n=0}^d$ we have:*

$$(i) \quad u_0 = \eta_k \frac{\langle \eta_i, \eta_\ell \rangle}{\langle \eta_k, \eta_\ell \rangle};$$

$$(ii) \quad u_d = \eta_\ell \frac{\langle \eta_k, \eta_i \rangle}{\langle \eta_k, \eta_\ell \rangle}.$$

Proof: (i) By construction there exists $\gamma \in \mathbb{F}$ such that $u_0 = \gamma \eta_k$. We show $\gamma = \langle \eta_i, \eta_\ell \rangle \langle \eta_k, \eta_\ell \rangle^{-1}$. By Lemma 11.5 we have $\langle u_n, \eta_\ell \rangle = 0$ for $1 \leq n \leq d$. By this and Lemma 12.2(ii),

$$\begin{aligned} \langle \eta_i, \eta_\ell \rangle &= \langle u_0 + \cdots + u_d, \eta_\ell \rangle \\ &= \langle u_0, \eta_\ell \rangle \\ &= \gamma \langle \eta_k, \eta_\ell \rangle. \end{aligned}$$

Therefore $\gamma = \langle \eta_i, \eta_\ell \rangle \langle \eta_k, \eta_\ell \rangle^{-1}$.

(ii) Similar to the proof of (i) above. \square

Proposition 12.5 *With reference to Notation 12.1, pick mutually distinct $i, j, k, \ell \in \mathbb{I}$ and consider the basis $[i, j, k, \ell]$ for V from Lemma 12.2. Denoting this basis by $\{u_n\}_{n=0}^d$ we have*

$$\langle u_r, u_s \rangle = \delta_{r+s,d} (-1)^r \binom{d}{r} \frac{\langle \eta_k, \eta_i \rangle \langle \eta_i, \eta_\ell \rangle}{\langle \eta_k, \eta_\ell \rangle} \quad (33)$$

for $0 \leq r, s \leq d$.

Proof: In line (31) evaluate $\langle u_0, u_d \rangle$ using Lemma 12.4. \square

13 Transition matrices between the 24 bases

In Lemma 12.2 we defined 24 bases for $V_d(a)$. In this section we compute the transition matrices between certain pairs of bases among these 24. First we recall a few terms.

Let V denote a vector space over \mathbb{F} with finite positive dimension. Suppose we are given two bases for V , written $\{u_n\}_{n=0}^d$ and $\{v_n\}_{n=0}^d$. By the *transition matrix* from $\{u_n\}_{n=0}^d$ to $\{v_n\}_{n=0}^d$ we mean the matrix $T \in \text{Mat}_{d+1}(\mathbb{F})$ such that

$$v_j = \sum_{i=0}^d T_{ij} u_i \quad (0 \leq j \leq d). \quad (34)$$

We recall a few properties of transition matrices. Let T denote the transition matrix from $\{u_n\}_{n=0}^d$ to $\{v_n\}_{n=0}^d$. Then T^{-1} exists, and equals the transition matrix from $\{v_n\}_{n=0}^d$ to $\{u_n\}_{n=0}^d$. Let $\{w_n\}_{n=0}^d$ denote a basis for V , and let S denote the transition matrix from $\{v_n\}_{n=0}^d$ to $\{w_n\}_{n=0}^d$. Then TS is the transition matrix from $\{u_n\}_{n=0}^d$ to $\{w_n\}_{n=0}^d$. For a linear transformation $A : V \rightarrow V$ let X (resp. Y) denote the matrix representing A with respect to $\{u_n\}_{n=0}^d$ (resp. $\{v_n\}_{n=0}^d$). Then $XT = TY$.

The following matrix will play a role in our discussion. For an integer $d \geq 0$ let $Z = Z(d)$ denote the matrix in $\text{Mat}_{d+1}(\mathbb{F})$ with entries

$$Z_{ij} = \begin{cases} 1, & \text{if } i + j = d; \\ 0, & \text{if } i + j \neq d \end{cases} \quad (0 \leq i, j \leq d). \quad (35)$$

We observe $Z^2 = I$.

Theorem 13.1 *With reference to Notation 12.1 and Lemma 12.2, pick mutually distinct $i, j, k, \ell \in \mathbb{I}$ and consider the transition matrices from the basis $[i, j, k, \ell]$ to the bases*

$$[j, i, k, \ell], \quad [i, k, j, \ell], \quad [i, j, \ell, k].$$

(i) *The first transition matrix is diagonal with (r, r) -entry*

$$\frac{\langle \eta_j, \eta_\ell \rangle}{\langle \eta_i, \eta_\ell \rangle} \alpha^r$$

for $0 \leq r \leq d$, where α is the (i, j, k, ℓ) -relative of a from Definition 9.1.

(ii) *The second transition matrix is lower triangular with (r, s) -entry*

$$\binom{r}{s} \alpha^{r-s} (1 - \alpha)^s$$

for $0 \leq s \leq r \leq d$, where α is the (i, j, k, ℓ) -relative of a from Definition 9.1.

(iii) *The third transition matrix is the matrix Z from (35).*

Proof: (i) Let T denote the transition matrix from $[i, j, k, \ell]$ to $[j, i, k, \ell]$. Then T is diagonal by Lemma 12.2(i). Let X (resp. Y) denote the matrix that represents $x_{j\ell}$ with respect to $[i, j, k, \ell]$ (resp. $[j, i, k, \ell]$). Then $XT = TY$. By Theorem 10.7, X is lower bidiagonal with entries $X_{nn} = 2n - d$ for $0 \leq n \leq d$ and $X_{n,n-1} = -2\alpha n$ for $1 \leq n \leq d$. Similarly Y is lower bidiagonal with entries $Y_{nn} = 2n - d$ for $0 \leq n \leq d$ and $Y_{n,n-1} = -2n$ for $1 \leq n \leq d$. Evaluating $XT = TY$ using these comments we find $T_{nn} = \alpha T_{n-1,n-1}$ for $1 \leq n \leq d$. Therefore

$$T_{rr} = T_{00} \alpha^r \quad (0 \leq r \leq d). \quad (36)$$

By Lemma 12.4(i) the 0th component of the basis $[i, j, k, \ell]$ is $\eta_k \langle \eta_i, \eta_\ell \rangle \langle \eta_k, \eta_\ell \rangle^{-1}$. Also by Lemma 12.4(i), the 0th component of the basis $[j, i, k, \ell]$ is $\eta_k \langle \eta_j, \eta_\ell \rangle \langle \eta_k, \eta_\ell \rangle^{-1}$. By these comments the 0th component of the basis $[j, i, k, \ell]$ is $\langle \eta_j, \eta_\ell \rangle \langle \eta_i, \eta_\ell \rangle^{-1}$ times the 0th component

of the basis $[i, j, k, \ell]$. Therefore $T_{00} = \langle \eta_j, \eta_\ell \rangle \langle \eta_i, \eta_\ell \rangle^{-1}$. Combining this with (36) we get the result.

(ii) Let T denote the transition matrix from $[i, j, k, \ell]$ to $[i, k, j, \ell]$. For each of these two bases the sum of the vectors is η_i , so T has constant row sum 1. By construction T is lower triangular. Let X (resp. Y) denote the matrix that represents $x_{\ell j}$ with respect to $[i, j, k, \ell]$ (resp. $[i, k, j, \ell]$). Then $XT = TY$. By Theorem 10.7, X is lower bidiagonal with entries $X_{nn} = d - 2n$ for $0 \leq n \leq d$ and $X_{n,n-1} = 2\alpha n$ for $1 \leq n \leq d$. Also Y is diagonal with entries $Y_{nn} = d - 2n$ for $0 \leq n \leq d$. Evaluating $XT = TY$ using these comments we find $(r - s)T_{rs} = \alpha r T_{r-1,s}$ for $0 \leq s < r \leq d$. By this recursion,

$$T_{rs} = \binom{r}{s} \alpha^{r-s} T_{ss} \quad (0 \leq s \leq r \leq d). \quad (37)$$

By (37) and since T has constant row sum 1 we routinely obtain $T_{ss} = (1 - \alpha)^s$ for $0 \leq s \leq d$ by induction on s . The result follows.

(iii) Immediate from Lemma 12.2. \square

With reference to Notation 12.1, we now consider how the scalars

$$\langle \eta_i, \eta_j \rangle \quad i, j \in \mathbb{I}, \quad i \neq j$$

are related.

Lemma 13.2 *With reference to Notation 12.1, for mutually distinct $i, j, k, \ell \in \mathbb{I}$ we have*

$$\frac{\langle \eta_i, \eta_\ell \rangle}{\langle \eta_i, \eta_k \rangle} \frac{\langle \eta_j, \eta_k \rangle}{\langle \eta_j, \eta_\ell \rangle} = \alpha^d,$$

where α is the (i, j, k, ℓ) -relative of a from Definition 9.1.

Proof: Let T (resp. T') denote the transition matrix from $[i, j, k, \ell]$ to $[j, i, k, \ell]$ (resp. $[j, i, \ell, k]$ to $[i, j, \ell, k]$). By Theorem 13.1(iii), the matrix Z is the transition matrix from $[j, i, k, \ell]$ to $[j, i, \ell, k]$, and from $[i, j, \ell, k]$ to $[i, j, k, \ell]$. Therefore $TZT'Z = I$. By Theorem 13.1(i), T is diagonal with entries $T_{nn} = \langle \eta_j, \eta_\ell \rangle \langle \eta_i, \eta_\ell \rangle^{-1} \alpha^n$ for $0 \leq n \leq d$. By Lemma 9.2(i),(iii) the (j, i, ℓ, k) -relative of a coincides with the (i, j, k, ℓ) -relative of a and is therefore equal to α . By this and Theorem 13.1(i), T' is diagonal with entries $T'_{nn} = \langle \eta_i, \eta_k \rangle \langle \eta_j, \eta_k \rangle^{-1} \alpha^n$ for $0 \leq n \leq d$. Evaluating $TZT'Z = I$ using these comments we obtain the result. \square

Corollary 13.3 *With reference to Notation 12.1 we have*

$$\begin{aligned} \frac{\langle \eta_0, \eta_1 \rangle}{\langle \eta_0, \eta_3 \rangle} \frac{\langle \eta_2, \eta_3 \rangle}{\langle \eta_2, \eta_1 \rangle} &= a^d, \\ \frac{\langle \eta_0, \eta_2 \rangle}{\langle \eta_0, \eta_1 \rangle} \frac{\langle \eta_3, \eta_1 \rangle}{\langle \eta_3, \eta_2 \rangle} &= (1 - a^{-1})^d, \\ \frac{\langle \eta_0, \eta_3 \rangle}{\langle \eta_0, \eta_2 \rangle} \frac{\langle \eta_1, \eta_2 \rangle}{\langle \eta_1, \eta_3 \rangle} &= (1 - a)^{-d}. \end{aligned}$$

Proof: Use Lemma 13.2 and Proposition 9.4. □

Note 13.4 By Corollary 13.3 and the symmetry/antisymmetry of $\langle \cdot, \cdot \rangle$ the scalars

$$\langle \eta_i, \eta_j \rangle \quad i, j \in \mathbb{I}, \quad i \neq j$$

are determined by the sequence

$$\langle \eta_0, \eta_1 \rangle, \quad \langle \eta_0, \eta_2 \rangle, \quad \langle \eta_0, \eta_3 \rangle, \quad \langle \eta_1, \eta_2 \rangle. \quad (38)$$

The scalars (38) are “free” in the following sense. Given a sequence Ψ of four nonzero scalars in \mathbb{F} , there exist vectors η_i ($i \in \mathbb{I}$) as in Notation 12.1 such that the sequence (38) is equal to Ψ .

14 Realizing the evaluation modules for \boxtimes using polynomials in two variables

In this section we give a concrete realization of the evaluation modules for \boxtimes using polynomials in two variables.

Notation 14.1 Let z_0, z_1 denote commuting indeterminates. Let $\mathbb{F}[z_0, z_1]$ denote the \mathbb{F} -algebra of all polynomials in z_0, z_1 that have coefficients in \mathbb{F} . We abbreviate $\mathcal{P} = \mathbb{F}[z_0, z_1]$ and often view this as a vector space over \mathbb{F} . For an integer $d \geq 0$ let \mathcal{P}_d denote the subspace of \mathcal{P} consisting of the homogeneous polynomials in z_0, z_1 that have total degree d . Thus $\{z_0^{d-n}z_1^n\}_{n=0}^d$ is a basis for \mathcal{P}_d . Note that

$$\mathcal{P} = \sum_{n=0}^{\infty} \mathcal{P}_n \quad (\text{direct sum}) \quad (39)$$

and that $\mathcal{P}_r\mathcal{P}_s = \mathcal{P}_{r+s}$ for $r, s \geq 0$. We fix mutually distinct $\beta_i \in \mathbb{F}$ ($i \in \mathbb{I}$). Then there exist unique $z_2, z_3 \in \mathcal{P}$ such that

$$\sum_{i \in \mathbb{I}} z_i = 0, \quad \sum_{i \in \mathbb{I}} \beta_i z_i = 0. \quad (40)$$

For the following three lemmas the proofs are routine and left to the reader.

Lemma 14.2 *With reference to Notation 14.1, for mutually distinct $i, j, k, \ell \in \mathbb{I}$ we have*

$$\begin{aligned} z_k &= \frac{\beta_\ell - \beta_i}{\beta_k - \beta_\ell} z_i + \frac{\beta_\ell - \beta_j}{\beta_k - \beta_\ell} z_j, \\ z_\ell &= \frac{\beta_i - \beta_k}{\beta_k - \beta_\ell} z_i + \frac{\beta_j - \beta_k}{\beta_k - \beta_\ell} z_j. \end{aligned}$$

Lemma 14.3 *With reference to Notation 14.1, for distinct $i, j \in \mathbb{I}$ the elements*

$$z_i^r z_j^s \quad 0 \leq r, s < \infty$$

form a basis for \mathcal{P} .

Lemma 14.4 *With reference to Notation 14.1, for an integer $d \geq 0$ and distinct $i, j \in \mathbb{I}$ the elements $\{z_i^{d-n} z_j^n\}_{n=0}^d$ form a basis for \mathcal{P}_d .*

Our next goal is to display a \boxtimes -module structure on \mathcal{P} . The following definition will be useful.

Definition 14.5 [17, p. 4] *With reference to Notation 14.1, by a *derivation* of \mathcal{P} we mean an \mathbb{F} -linear map $D : \mathcal{P} \rightarrow \mathcal{P}$ such that $D(uv) = D(u)v + uD(v)$ for $u, v \in \mathcal{P}$.*

We have a comment.

Lemma 14.6 *With reference to Notation 14.1 let D denote a derivation of \mathcal{P} . Then $D(z^r) = rz^{r-1}D(z)$ for $z \in \mathcal{P}$ and $r \geq 1$. Moreover $D(1) = 0$.*

Proof: The first equation is a routine consequence of Definition 14.5. Setting $z = 1$ and $r = 2$ in this equation we find $D(1) = 2D(1)$ so $D(1) = 0$. \square

Lemma 14.7 *With reference to Notation 14.1, a derivation of \mathcal{P} is 0 if and only if it vanishes on \mathcal{P}_1 .*

Proof: Let D denote the derivation in question. We assume D vanishes on \mathcal{P}_1 and show $D = 0$. For distinct $i, j \in \mathbb{I}$ and integers $r, s \geq 0$ we have $D(z_i^r z_j^s) = 0$ by Definition 14.5 and Lemma 14.6. Now $D = 0$ in view of Lemma 14.3. \square

Proposition 14.8 *With reference to Notation 14.1, there exists a unique \boxtimes -module structure on \mathcal{P} that satisfies (i), (ii) below:*

(i) *each element of \boxtimes acts as a derivation on \mathcal{P} ;*

(ii) *$x_{ij}.z_i = -z_i$ and $x_{ij}.z_j = z_j$ for distinct $i, j \in \mathbb{I}$.*

Proof: Let $\text{Der } \mathcal{P}$ denote the set of derivations for \mathcal{P} . Recall that $\text{Der } \mathcal{P}$ is a Lie algebra over \mathbb{F} with Lie bracket $[D, D'] = DD' - D'D$. For distinct $i, j \in \mathbb{I}$ define an \mathbb{F} -linear map $D_{ij} : \mathcal{P} \rightarrow \mathcal{P}$ such that

$$D_{ij}(z_i^r z_j^s) = (s - r)z_i^r z_j^s \quad r, s = 0, 1, 2, \dots$$

One checks that D_{ij} is the unique element of $\text{Der } \mathcal{P}$ that sends $z_i \mapsto -z_i$ and $z_j \mapsto z_j$. Next one checks that the maps $\{D_{ij} | i, j \in \mathbb{I}, i \neq j\}$ satisfy the defining relations for \boxtimes given in Definition 2.2. To do this, it suffices to verify that these relations hold on \mathcal{P}_1 , in view of Lemma 14.7. From our comments so far, there exists a Lie algebra homomorphism from \boxtimes to $\text{Der } \mathcal{P}$ that sends $x_{ij} \mapsto D_{ij}$ for all distinct $i, j \in \mathbb{I}$. This shows that the required \boxtimes -module structure exists. This \boxtimes -module structure is unique in view of Lemma 14.7. \square

We emphasize the following.

Lemma 14.9 *With reference to Notation 14.1 and Proposition 14.8, for distinct $i, j \in \mathbb{I}$ and integers $r, s \geq 0$ the element $z_i^r z_j^s$ is an eigenvector for x_{ij} with eigenvalue $s - r$.*

Lemma 14.10 *With reference to Notation 14.1 the following (i)–(iii) hold.*

- (i) *For $d \geq 0$ the subspace \mathcal{P}_d is an irreducible \boxtimes -submodule of \mathcal{P} .*
- (ii) *The \boxtimes -module \mathcal{P}_0 is trivial.*
- (iii) *For $d \geq 1$ the \boxtimes -module \mathcal{P}_d is isomorphic to an evaluation module.*

Proof: Pick distinct $i, j \in \mathbb{I}$ and consider the basis $\{z_i^{d-n} z_j^n\}_{n=0}^d$ for \mathcal{P}_d from Lemma 14.4. By Lemma 14.9, for $0 \leq n \leq d$ the vector $z_i^{d-n} z_j^n$ is an eigenvector for x_{ij} with eigenvalue $2n - d$. Therefore \mathcal{P}_d is invariant under x_{ij} . We have now shown that \mathcal{P}_d is a \boxtimes -submodule. This module is irreducible since the eigenvalues of x_{ij} on \mathcal{P}_d are $2n - d$ ($0 \leq n \leq d$) and the corresponding eigenspaces have dimension 1. This proves (i). To get (ii) note that \mathcal{P}_0 has dimension 1. To get (iii) apply Proposition 6.4. \square

We now find the evaluation parameter for the \boxtimes -modules given in Lemma 14.10(iii). The following lemma will be useful.

Lemma 14.11 *With reference to Notation 14.1, for mutually distinct $i, j, k, \ell \in \mathbb{I}$ the following vanishes on the \boxtimes -module \mathcal{P} :*

$$(\beta_i - \beta_j)(\beta_k - \beta_\ell)x_{ij} + (\beta_i - \beta_k)(\beta_\ell - \beta_j)x_{ik} + (\beta_i - \beta_\ell)(\beta_j - \beta_k)x_{i\ell}. \quad (41)$$

Proof: Using Lemma 14.2 and Proposition 14.8(ii) we routinely find that the derivation (41) vanishes on each of z_i, z_j and hence on \mathcal{P}_1 . This derivation is 0 in view of Lemma 14.7. \square

Theorem 14.12 *With reference to Notation 14.1, for an integer $d \geq 1$ the \boxtimes -module \mathcal{P}_d is isomorphic to $V_d(a)$ where*

$$a = \frac{\beta_0 - \beta_1}{\beta_0 - \beta_3} \frac{\beta_2 - \beta_3}{\beta_2 - \beta_1}.$$

Proof: We invoke Proposition 6.1 with $V = \mathcal{P}_d$. Note that $a \notin \{0, 1\}$ since the β_i ($i \in \mathbb{I}$) are mutually distinct. By Lemma 14.11, the four expressions in (18), (19) vanish on \mathcal{P} and in particular on \mathcal{P}_d . By this and Proposition 6.1 the \boxtimes -module \mathcal{P}_d has evaluation parameter a . Now the \boxtimes -module \mathcal{P}_d is isomorphic to $V_d(a)$ since \mathcal{P}_d has dimension $d + 1$. \square

Earlier in the paper we described the \boxtimes -module $V_d(a)$. We now consider how things look from the point of view of \mathcal{P}_d .

Proposition 14.13 *With reference to Notation 14.1, for an integer $d \geq 0$ and distinct $i, j \in \mathbb{I}$ the decomposition $[i, j]$ on \mathcal{P}_d is described as follows: for $0 \leq n \leq d$ the n th component is spanned by $z_i^{d-n} z_j^n$.*

Proof: By Lemma 14.4 the vectors $\{z_i^{d-n} z_j^n\}_{n=0}^d$ form a basis for \mathcal{P}_d . By Lemma 14.9, for $0 \leq n \leq d$ the vector $z_i^{d-n} z_j^n$ is an eigenvector for x_{ij} with eigenvalue $2n - d$. The result follows. \square

Proposition 14.14 *With reference to Notation 14.1, for an integer $d \geq 0$ and $i \in \mathbb{I}$ the flag $[i]$ on \mathcal{P}_d is described as follows: for $0 \leq n \leq d$ the n th component is $z_i^{d-n} \mathcal{P}_n$.*

Proof: Let U_n denote the component in question and pick $j \in \mathbb{I}, j \neq i$. By construction U_n is the sum of components $0, 1, \dots, n$ for the decomposition $[i, j]$ of \mathcal{P}_d . By this and Proposition 14.13, U_n has a basis $\{z_i^{d-r} z_j^r\}_{r=0}^n$. Note that $\{z_i^{n-r} z_j^r\}_{r=0}^n$ is a basis for \mathcal{P}_n so $\{z_i^{d-r} z_j^r\}_{r=0}^n$ is a basis for $z_i^{d-n} \mathcal{P}_n$. Therefore $U_n = z_i^{d-n} \mathcal{P}_n$. \square

Definition 14.15 *With reference to Notation 14.1, for an integer $d \geq 1$ and $i \in \mathbb{I}$ the element z_i^d is in component 0 of the flag $[i]$ on \mathcal{P}_d . In Notation 12.1 we let η_i denote any nonzero element in this component. For the rest of this section we choose $\eta_i = z_i^d$.*

Proposition 14.16 *With reference to Notation 14.1, for an integer $d \geq 1$ and for mutually distinct $i, j, k, \ell \in \mathbb{I}$ the basis $[i, j, k, \ell]$ of \mathcal{P}_d is described as follows. For $0 \leq n \leq d$ the n th component is*

$$z_k^{d-n} z_\ell^n \binom{d}{n} \frac{(\beta_j - \beta_k)^{d-n} (\beta_j - \beta_\ell)^n}{(\beta_i - \beta_j)^d}. \quad (42)$$

Proof: Abbreviate u_n for the polynomial (42) and note that $\{u_n\}_{n=0}^d$ is a basis for \mathcal{P}_d by Lemma 14.4. We show that this basis satisfies Lemma 12.2(i),(ii). By Proposition 14.13, for $0 \leq n \leq d$ the vector u_n is in component n of the decomposition $[k, \ell]$ for \mathcal{P}_d . Also $\eta_i = z_i^d$ by Definition 14.15, and

$$\begin{aligned} z_i^d &= \left(\frac{\beta_j - \beta_k}{\beta_i - \beta_j} z_k + \frac{\beta_j - \beta_\ell}{\beta_i - \beta_j} z_\ell \right)^d \\ &= \sum_{n=0}^d u_n \end{aligned}$$

by Lemma 14.2 and the binomial theorem. Therefore the basis $\{u_n\}_{n=0}^d$ satisfies Lemma 12.2(i),(ii) and the result follows. \square

We have a comment on the S_4 action.

Lemma 14.17 *With reference to Notation 14.1 and Theorem 14.12, for mutually distinct $i, j, k, \ell \in \mathbb{I}$ the (i, j, k, ℓ) -relative of a is*

$$\frac{\beta_i - \beta_\ell}{\beta_i - \beta_k} \frac{\beta_j - \beta_k}{\beta_j - \beta_\ell}.$$

Proof: Very similar to the proof of Lemma 9.3. \square

We now consider the standard bilinear form.

Lemma 14.18 *With reference to Notation 14.1, for an integer $d \geq 1$ there exists a standard bilinear form \langle , \rangle on the \boxtimes -module \mathcal{P}_d that satisfies the following. For distinct $i, j \in \mathbb{I}$,*

$$\langle z_i^d, z_j^d \rangle = (\beta_k - \beta_\ell)^d$$

where the set $\{k, \ell\}$ is the complement of $\{i, j\}$ in \mathbb{I} , and the pair k, ℓ is ordered so that the sequence (i, j, k, ℓ) is sent to $(2, 0, 1, 3)$ by an even permutation in S_4 .

Proof: For the time being let \langle , \rangle denote any standard bilinear form on \mathcal{P}_d . For mutually distinct $i, j, k, \ell \in \mathbb{I}$ define

$$\tau(i, j, k, \ell) = \frac{\langle z_i^d, z_j^d \rangle}{(\beta_k - \beta_\ell)^d},$$

and note that $\tau(i, j, k, \ell) \neq 0$ by Lemma 12.3(ii). We claim

- $\tau(j, i, k, \ell) = (-1)^d \tau(i, j, k, \ell)$,
- $\tau(i, k, j, \ell) = (-1)^d \tau(i, j, k, \ell)$,
- $\tau(i, j, \ell, k) = (-1)^d \tau(i, j, k, \ell)$.

The first bullet follows from the symmetry/antisymmetry of \langle , \rangle in Lemma 11.2. The third bullet follows from the construction. To verify the second bullet we show

$$\frac{\langle z_i^d, z_k^d \rangle}{(\beta_\ell - \beta_j)^d} = \frac{\langle z_i^d, z_j^d \rangle}{(\beta_k - \beta_\ell)^d}. \quad (43)$$

To verify (43), in the left-hand side eliminate z_k using the first equation of Lemma 14.2, and note that $\langle z_i^d, z_i^{d-n} z_j^n \rangle = 0$ for $0 \leq n \leq d-1$ in view of Lemma 11.5 and Proposition 14.13. This proves (43) and we have now verified the three bullets. Multiplying \langle , \rangle by a nonzero scalar in \mathbb{F} if necessary, we may assume $\tau(2, 0, 1, 3) = 1$. For $\sigma \in S_4$ define $\text{sgn}(\sigma)$ to be 1 if σ is even and -1 if σ is odd. Using the three bullets and $\tau(2, 0, 1, 3) = 1$ we obtain

$$\tau(i, j, k, \ell) = \text{sgn}(\sigma)^d$$

where $\sigma \in S_4$ sends the sequence (i, j, k, ℓ) to $(2, 0, 1, 3)$. The result follows. \square

Corollary 14.19 *Referring to the bilinear form \langle , \rangle in Lemma 14.18, for $i, j \in \mathbb{I}$ the scalar $\langle z_i^d, z_j^d \rangle$ is displayed in row i , column j of the table below.*

| | 0 | 1 | 2 | 3 |
|---|-------------------------|-------------------------|-------------------------|-------------------------|
| 0 | 0 | $(\beta_2 - \beta_3)^d$ | $(\beta_3 - \beta_1)^d$ | $(\beta_1 - \beta_2)^d$ |
| 1 | $(\beta_3 - \beta_2)^d$ | 0 | $(\beta_0 - \beta_3)^d$ | $(\beta_2 - \beta_0)^d$ |
| 2 | $(\beta_1 - \beta_3)^d$ | $(\beta_3 - \beta_0)^d$ | 0 | $(\beta_0 - \beta_1)^d$ |
| 3 | $(\beta_2 - \beta_1)^d$ | $(\beta_0 - \beta_2)^d$ | $(\beta_1 - \beta_0)^d$ | 0 |

Proof: Routine using Lemma 12.3(i) and Lemma 14.18. \square

Theorem 14.20 *With reference to Notation 14.1, for an integer $d \geq 1$ consider the standard bilinear form $\langle \cdot, \cdot \rangle$ on \mathcal{P}_d from Lemma 14.18. For distinct $i, j \in \mathbb{I}$ and $0 \leq r, s \leq d$ we have*

$$\langle z_i^{d-r} z_j^r, z_i^{d-s} z_j^s \rangle = \delta_{r+s,d} (-1)^r \binom{d}{r}^{-1} (\beta_k - \beta_\ell)^d \quad (44)$$

where the set $\{k, \ell\}$ is the complement of $\{i, j\}$ in \mathbb{I} , and the pair k, ℓ is ordered so that the sequence (i, j, k, ℓ) is sent to $(2, 0, 1, 3)$ by an even permutation in S_4 .

Proof: Assume $r + s = d$; otherwise (44) holds by Lemma 11.5 and Proposition 14.13. Let $\{u_n\}_{n=0}^d$ denote the basis $[k, \ell, i, j]$ for \mathcal{P}_d . Using Proposition 11.7 and Proposition 14.16 we find

$$\begin{aligned} (-1)^r \binom{d}{r} &= \frac{\langle u_r, u_{d-r} \rangle}{\langle u_0, u_d \rangle} \\ &= \frac{\langle z_i^{d-r} z_j^r, z_i^r z_j^{d-r} \rangle}{\langle z_i^d, z_j^d \rangle} \binom{d}{r}^2. \end{aligned}$$

Line (44) follows in view of Lemma 14.18. \square

Recall the subgroup G of S_4 from Definition 8.1. In Corollary 8.3 we showed that for an evaluation module of \boxtimes and for $\sigma \in G$, twisting the module via σ does not change the isomorphism class of the module. We now interpret this fact using the \boxtimes -module \mathcal{P} .

With reference to Notation 14.1, recall an *automorphism* of \mathcal{P} is an \mathbb{F} -linear bijection $\phi : \mathcal{P} \rightarrow \mathcal{P}$ such that $\phi(uv) = \phi(u)\phi(v)$ for $u, v \in \mathcal{P}$.

Lemma 14.21 *With reference to Notation 14.1, for an automorphism ϕ of \mathcal{P} and a derivation D of \mathcal{P} the composition $\phi^{-1}D\phi$ is a derivation of \mathcal{P} .*

Proof: Routine using Definition 14.5 and the definition of automorphism. \square

Lemma 14.22 *With reference to Notation 14.1, for mutually distinct $i, j, k, \ell \in \mathbb{I}$ there exists a unique automorphism of \mathcal{P} that sends*

$$z_i \mapsto \frac{\beta_j - \beta_k}{\beta_i - \beta_k} z_j, \quad z_j \mapsto \frac{\beta_i - \beta_\ell}{\beta_j - \beta_\ell} z_i, \quad (45)$$

$$z_k \mapsto \frac{\beta_\ell - \beta_i}{\beta_i - \beta_k} z_\ell, \quad z_\ell \mapsto \frac{\beta_k - \beta_j}{\beta_j - \beta_\ell} z_k. \quad (46)$$

Proof: Since z_i, z_j form a basis for \mathcal{P}_1 there exists an \mathbb{F} -linear transformation $\phi : \mathcal{P}_1 \rightarrow \mathcal{P}_1$ that satisfies (45). Observe that ϕ^{-1} exists, and that ϕ extends uniquely to an automorphism of \mathcal{P} . To get (46) combine Lemma 14.2 and (45). \square

Proposition 14.23 *With reference to Notation 14.1, for $\sigma \in G$ there exists an automorphism ϕ_σ of \mathcal{P} that sends z_r into $\mathbb{F}z_{\sigma(r)}$ for all $r \in \mathbb{I}$.*

Proof: Assume σ is not the identity; otherwise the result is clear. By Definition 8.1 there exists mutually distinct $i, j, k, \ell \in \mathbb{I}$ such that $\sigma = (i, j)(k, \ell)$; let ϕ_σ denote the corresponding automorphism of \mathcal{P} from Lemma 14.22. By (45), (46) the automorphism ϕ_σ sends z_r into $\mathbb{F}z_{\sigma(r)}$ for all $r \in \mathbb{I}$. \square

Proposition 14.24 *With reference to Notation 14.1, for $\sigma \in G$ and $\xi \in \boxtimes$ the equation*

$$\sigma(\xi) = \phi_\sigma \xi \phi_\sigma^{-1} \quad (47)$$

holds on \mathcal{P} .

Proof: Without loss we may assume that ξ is a generator x_{rs} of \boxtimes . Each side of (47) acts on \mathcal{P} as a derivation, by Proposition 14.8 and Lemma 14.21. Now by Lemma 14.7, it suffices to show that (47) holds on \mathcal{P}_1 . The elements $z_{\sigma(r)}$ and $z_{\sigma(s)}$ form a basis for \mathcal{P}_1 . We now apply each side of (47) to $z_{\sigma(r)}$. Concerning the left-hand side,

$$\begin{aligned} \sigma(x_{rs}) \cdot z_{\sigma(r)} &= x_{\sigma(r), \sigma(s)} \cdot z_{\sigma(r)} \\ &= -z_{\sigma(r)}. \end{aligned}$$

Concerning the right-hand side, first note by Proposition 14.23 that there exists $\gamma \in \mathbb{F}$ such that $\phi_\sigma(z_r) = \gamma z_{\sigma(r)}$. Observe $\gamma \neq 0$ since ϕ_σ is a bijection. We have

$$\begin{aligned} \phi_\sigma x_{rs} \phi_\sigma^{-1} \cdot z_{\sigma(r)} &= \gamma^{-1} \phi_\sigma x_{rs} \cdot z_r \\ &= -\gamma^{-1} \phi_\sigma \cdot z_r \\ &= -z_{\sigma(r)}. \end{aligned}$$

Thus the two sides of (47) agree at $z_{\sigma(r)}$. A similar argument shows that the two sides of (47) agree at $z_{\sigma(s)}$. Now (47) holds on \mathcal{P}_1 and the result follows. \square

Theorem 14.25 *With reference to Notation 14.1 and Proposition 14.23, for $\sigma \in G$ the map ϕ_σ is an isomorphism of \boxtimes -modules from \mathcal{P} to \mathcal{P} twisted via σ .*

Proof: This is a reformulation of Proposition 14.24. \square

15 Finite-dimensional irreducible \boxtimes -modules

We have now completed our description of the evaluation modules for \boxtimes . For the rest of this paper we consider general finite-dimensional irreducible \boxtimes -modules. For these we will not go into as much detail as we did for the evaluation modules, but there are a few points we would like to make.

We start with some comments on tensor products.

Let U and V denote \boxtimes -modules. By [17, p. 26] the vector space $U \otimes V$ has a \boxtimes -module structure given by

$$\xi.(u \otimes v) = (\xi.u) \otimes v + u \otimes (\xi.v) \quad u \in U, \quad v \in V, \quad \xi \in \boxtimes. \quad (48)$$

Lemma 15.1 *The following (i), (ii) hold for \boxtimes -modules U and V :*

(i) *There exists an isomorphism of \boxtimes -modules from $U \otimes V$ to $V \otimes U$ that sends $u \otimes v \mapsto v \otimes u$ for all $u \in U$ and $v \in V$.*

(ii) *Assume the \boxtimes -module $U \otimes V$ is irreducible. Then U and V are irreducible.*

Proof: (i) Routine.

(ii) The \boxtimes -module U is irreducible since if U' is a nonzero proper submodule of U then $U' \otimes V$ is a nonzero proper submodule of $U \otimes V$. The proof for V is similar. \square

Theorem 15.2 [15, Section 1] *Every nontrivial finite-dimensional irreducible \boxtimes -module is isomorphic to a tensor product of evaluation modules. Two such tensor products are isomorphic if and only if one can be obtained from the other by permuting the factors in the tensor product. A tensor product of evaluation modules*

$$V_{d_1}(a_1) \otimes V_{d_2}(a_2) \otimes \cdots \otimes V_{d_N}(a_N)$$

is irreducible if and only if a_1, a_2, \dots, a_N are mutually distinct.

Proof: In [15, Section 1] Hartwig considers a certain Lie algebra \mathcal{O} called the *Onsager algebra* [30], [32], [42]. In [15, Theorems 1.7, 1.8] he gives an explicit bijection between (i) the set of isomorphism classes of finite-dimensional irreducible \boxtimes -modules; (ii) the set of isomorphism classes of finite-dimensional irreducible \mathcal{O} -modules that have type $(0, 0)$. In [15, Theorems 1.3, 1.4, 1.6] Hartwig summarizes the classification of finite-dimensional irreducible \mathcal{O} -modules that have type $(0, 0)$. This classification is due to Davies [10], [11]; see also Date and Roan [9]. If we interpret this classification as a classification of the finite-dimensional irreducible \boxtimes -modules via the above bijection, we get the present theorem. To aid in this interpretation we note that our definition of the evaluation parameter does not match Hartwig's definition of the evaluation parameter. Denoting our evaluation parameter by a and Hartwig's evaluation parameter by b we have $a = 4b(b+1)^{-2}$. \square

We have some comments.

Lemma 15.3 *Let U and V denote \boxtimes -modules. Then for $\sigma \in S_4$ the following coincide:*

(i) *the \boxtimes -module ${}^\sigma(U \otimes V)$;*

(ii) *the \boxtimes -module ${}^\sigma U \otimes {}^\sigma V$.*

Proof: Routine using Definition 7.1 and (48). \square

Theorem 15.4 *Let V denote a nontrivial finite-dimensional irreducible \boxtimes -module, and write V as a tensor product of evaluation modules:*

$$V = V_{d_1}(a_1) \otimes V_{d_2}(a_2) \otimes \cdots \otimes V_{d_N}(a_N).$$

Then for $\sigma \in S_4$ the \boxtimes -module ${}^\sigma V$ is isomorphic to

$$V_{d_1}(\sigma(a_1)) \otimes V_{d_2}(\sigma(a_2)) \otimes \cdots \otimes V_{d_N}(\sigma(a_N)).$$

Proof: Combine Theorem 7.7 and Lemma 15.3. \square

Corollary 15.5 *Let V denote a finite-dimensional irreducible \boxtimes -module and let the group G be as in Definition 8.1. Then for $\sigma \in G$ the following are isomorphic:*

- (i) *the \boxtimes -module V twisted via σ ;*
- (ii) *the \boxtimes -module V .*

Proof: Combine Lemma 8.2 and Theorem 15.4. \square

16 Finite-dimensional irreducible \boxtimes -modules; the shape polynomial

In this section we obtain an explicit formula for the shape of a finite-dimensional irreducible \boxtimes -module. We use the following notation. For an indeterminate λ let $\mathbb{F}[\lambda]$ denote the \mathbb{F} -algebra consisting of all polynomials in λ that have coefficients in \mathbb{F} .

Definition 16.1 Let V denote a finite-dimensional irreducible \boxtimes -module. We define a polynomial $S_V \in \mathbb{F}[\lambda]$ by

$$S_V = \sum_{n=0}^d \rho_n \lambda^n,$$

where $\{\rho_n\}_{n=0}^d$ is the shape of V . We call S_V the *shape polynomial* for V .

Example 16.2 *Let V denote an evaluation module for \boxtimes . Then*

$$S_V = 1 + \lambda + \lambda^2 + \cdots + \lambda^d$$

where d is the diameter of V .

Proof: Combine Proposition 6.4 and Definition 16.1. \square

Let U, V denote finite-dimensional irreducible \boxtimes -modules such that the \boxtimes -module $U \otimes V$ is irreducible. We are going to show that $S_{U \otimes V} = S_U S_V$. We will use the following lemma.

Lemma 16.3 *Let U, V denote finite-dimensional irreducible \boxtimes -modules such that the \boxtimes -module $U \otimes V$ is irreducible. Let d (resp. δ) denote the diameter of U (resp. V).*

(i) *The diameter of $U \otimes V$ is $d + \delta$.*

(ii) *For distinct $i, j \in \mathbb{I}$ the decomposition $[i, j]$ of $U \otimes V$ is described as follows. For $0 \leq n \leq d + \delta$ the n th component is*

$$\sum_{r,s} U_r \otimes V_s, \quad (49)$$

where $\{U_r\}_{r=0}^d$ (resp. $\{V_s\}_{s=0}^\delta$) denotes the decomposition $[i, j]$ of U (resp. V), and where the sum is over all ordered pairs r, s such that $0 \leq r \leq d$, $0 \leq s \leq \delta$, $r + s = n$.

Proof: For $0 \leq n \leq d + \delta$ let $(U \otimes V)_n$ denote the sum (49). By (48), each element of $(U \otimes V)_n$ is an eigenvector for x_{ij} with eigenvalue $2n - d - \delta$. By construction the sequence $\{(U \otimes V)_n\}_{n=0}^{d+\delta}$ is a decomposition of $U \otimes V$. The results follow. \square

Corollary 16.4 *Let U, V denote finite-dimensional irreducible \boxtimes -modules such that the \boxtimes -module $U \otimes V$ is irreducible. Then with reference to Definition 16.1,*

$$S_{U \otimes V} = S_U S_V.$$

Proof: Adopt the notation of Lemma 16.3. For $0 \leq n \leq d + \delta$ the sum (49) is direct so

$$\rho_n(U \otimes V) = \sum_{r,s} \rho_r(U) \rho_s(V), \quad (50)$$

where the sum is over all ordered pairs r, s such that $0 \leq r \leq d$, $0 \leq s \leq \delta$, $r + s = n$. By Definition 16.1,

$$S_{U \otimes V} = \sum_{n=0}^{d+\delta} \rho_n(U \otimes V) \lambda^n.$$

Evaluating this using (50) we routinely find $S_{U \otimes V} = S_U S_V$. \square

Theorem 16.5 *Let V denote a nontrivial finite-dimensional irreducible \boxtimes -module, and write V as a tensor product of evaluation modules:*

$$V = V_{d_1}(a_1) \otimes V_{d_2}(a_2) \otimes \cdots \otimes V_{d_N}(a_N).$$

Then the shape $\{\rho_n\}_{n=0}^d$ of V satisfies

$$\sum_{n=0}^d \rho_n \lambda^n = \prod_{j=1}^N (1 + \lambda + \lambda^2 + \cdots + \lambda^{d_j}).$$

In particular $\rho_0 = 1$.

Proof: Combine Example 16.2 and Corollary 16.4. \square

17 Finite-dimensional irreducible \boxtimes -modules; the Drinfel'd polynomial

Let V denote a finite-dimensional irreducible \boxtimes -module. In this section we define a certain polynomial P_V called the Drinfel'd polynomial. We show that the map $V \mapsto P_V$ induces a bijection between the following two sets: (i) the isomorphism classes of finite-dimensional irreducible \boxtimes -modules; (ii) the polynomials in $\mathbb{F}[\lambda]$ that have constant coefficient 1 and are nonzero at $\lambda = 1$. We begin with a comment.

Lemma 17.1 *Let V denote a finite-dimensional irreducible \boxtimes -module. Pick mutually distinct $i, j, k \in \mathbb{I}$ and consider the action of $x_{ij} + x_{jk}$ on the decomposition $[i, j]$ of V . Denoting this decomposition by $\{V_n\}_{n=0}^d$ we have $(x_{ij} + x_{jk})V_n \subseteq V_{n+1}$ for $0 \leq n \leq d$.*

Proof: By the definition of $[i, j]$,

$$(x_{ij} - (2n - d)I)V_n = 0. \quad (51)$$

By the table in Section 3,

$$(x_{jk} - (d - 2n)I)V_n \subseteq V_{n+1}. \quad (52)$$

Adding (51), (52) we get the result. \square

Definition 17.2 Let V denote a finite-dimensional irreducible \boxtimes -module and let $\{V_n\}_{n=0}^d$ denote the decomposition $[1, 3]$ of V . Note that $\dim(V_0) = 1$ by Theorem 16.5. Abbreviate

$$e^+ := \frac{x_{13} + x_{30}}{2}, \quad e^- := \frac{x_{31} + x_{12}}{2}. \quad (53)$$

By Lemma 17.1 and since the decomposition $[3, 1]$ is the inversion of $[1, 3]$,

$$e^+V_n \subseteq V_{n+1}, \quad e^-V_n \subseteq V_{n-1} \quad (0 \leq n \leq d). \quad (54)$$

For an integer $i \geq 0$ the space V_0 is invariant under $(e^-)^i(e^+)^i$; let $\vartheta_i = \vartheta_i(V)$ denote the corresponding eigenvalue. Note that $\vartheta_i = 0$ for $i > d$.

Definition 17.3 Let V denote a finite-dimensional irreducible \boxtimes -module. We define a polynomial $P_V \in \mathbb{F}[\lambda]$ by

$$P_V = \sum_{i=0}^{\infty} \frac{(-1)^i \vartheta_i \lambda^i}{(i!)^2}, \quad (55)$$

where the scalars ϑ_i are from Definition 17.2. Observe that P_V has degree at most the diameter of V . Moreover P_V has constant coefficient $\vartheta_0 = 1$. Following [6, Section 3.4], [22, Definition 4.2], [26, Section 4] we call P_V the *Drinfel'd polynomial* of V .

Example 17.4 Let $V = V_d(a)$ denote an evaluation module for \boxtimes . Then $P_V = (1 - a\lambda)^d$.

Proof: Let $\{V_n\}_{n=0}^d$ denote the decomposition $[1, 3]$ of V . Let $\{v_n\}_{n=0}^d$ denote the basis for the \mathfrak{sl}_2 -module V_d from Lemma 5.5. Note that v_n is a basis of V_n for $0 \leq n \leq d$. Using (3), (53) and Lemma 5.3 we find $EV_a(e^+) = af$ and $EV_a(e^-) = e$. By this and Lemma 5.5 we find $e^+.v_n = a(n+1)v_{n+1}$ for $0 \leq n \leq d-1$, $e^+.v_d = 0$, $e^-.v_n = (d-n+1)v_{n-1}$ for $1 \leq n \leq d$, $e^-.v_0 = 0$. Using this data we find $\vartheta_i = a^i(i!)^2 \binom{d}{i}$ for $0 \leq i \leq d$. Now (55) becomes

$$\begin{aligned} P_V &= \sum_{i=0}^d \binom{d}{i} (-1)^i a^i \lambda^i \\ &= (1 - a\lambda)^d \end{aligned}$$

by the binomial theorem. □

The Drinfel'd polynomial has the following property.

Proposition 17.5 Let U, V denote finite-dimensional irreducible \boxtimes -modules such that the \boxtimes -module $U \otimes V$ is irreducible. Then

$$P_{U \otimes V} = P_U P_V.$$

Proof: We claim that for an integer $i \geq 0$,

$$\vartheta_i(U \otimes V) = \sum_{n=0}^i \binom{i}{n}^2 \vartheta_{i-n}(U) \vartheta_n(V). \quad (56)$$

To prove the claim, let U_0 (resp. V_0) denote the 0th component for the decomposition $[1, 3]$ of U (resp. V). By Theorem 16.5, each of U_0, V_0 has dimension 1. By Lemma 16.3(ii) the space $U_0 \otimes V_0$ is the 0th component of the decomposition $[1, 3]$ of $U \otimes V$. Therefore by Definition 17.2 the scalar $\vartheta_i(U \otimes V)$ is the eigenvalue of $(e^-)^i (e^+)^i$ associated with $U_0 \otimes V_0$. Pick $0 \neq u \in U_0$ and $0 \neq v \in V_0$. Using (48) we obtain

$$\begin{aligned} (e^-)^i (e^+)^i . (u \otimes v) &= (e^-)^i \sum_{n=0}^i \binom{i}{n} ((e^+)^{i-n} . u) \otimes ((e^+)^n . v) \\ &= \sum_{m=0}^i \sum_{n=0}^i \binom{i}{m} \binom{i}{n} ((e^-)^{i-m} (e^+)^{i-n} . u) \otimes ((e^-)^m (e^+)^n . v). \end{aligned} \quad (57)$$

We examine the terms in (57). By (54) and the line below it, for $0 \leq m, n \leq i$ the vector $(e^-)^m (e^+)^n . v$ is equal to 0 if $m > n$ and $\vartheta_n(V)v$ if $m = n$. Similarly $(e^-)^{i-m} (e^+)^{i-n} . u$ is equal to 0 if $m < n$ and $\vartheta_{i-n}(U)u$ if $m = n$. By these comments the double sum in (57) is equal to $u \otimes v$ times the sum on the right in (56). We conclude that (56) is valid and the claim is proved. By Definition 17.3,

$$P_{U \otimes V} = \sum_{i=0}^{\infty} \frac{(-1)^i \vartheta_i(U \otimes V) \lambda^i}{(i!)^2}.$$

Evaluating this using (56) we get $P_{U \otimes V} = P_U P_V$ after a brief calculation. □

Theorem 17.6 *Let V denote a nontrivial finite-dimensional irreducible \boxtimes -module, and write V as a tensor product of evaluation modules:*

$$V = V_{d_1}(a_1) \otimes V_{d_2}(a_2) \otimes \cdots \otimes V_{d_N}(a_N).$$

Then the Drinfel'd polynomial P_V is given by

$$P_V = \prod_{j=1}^N (1 - a_j \lambda)^{d_j}.$$

Proof: Combine Example 17.4 and Proposition 17.5. □

Corollary 17.7 *The map $V \mapsto P_V$ induces a bijection between the following two sets:*

- (i) *the isomorphism classes of finite-dimensional irreducible \boxtimes -modules;*
- (ii) *the polynomials in $\mathbb{F}[\lambda]$ that have constant coefficient 1 and are nonzero at $\lambda = 1$.*

Proof: Combine Theorem 15.2 and Theorem 17.6. □

18 Finite-dimensional irreducible \boxtimes -modules; the dual space

In this section we show that a finite-dimensional irreducible \boxtimes -module is isomorphic to its dual.

Theorem 18.1 *Let V denote a finite-dimensional irreducible \boxtimes -module. Then there exists a nonzero bilinear form $\langle \cdot, \cdot \rangle$ on V such that*

$$\langle \xi.u, v \rangle = -\langle u, \xi.v \rangle \quad \xi \in \boxtimes, \quad u, v \in V. \quad (58)$$

This form is unique up to multiplication by a nonzero scalar in \mathbb{F} . The form is nondegenerate. The form is symmetric (resp. antisymmetric) when the diameter is even (resp. odd).

Proof: Assume V is nontrivial; otherwise the proof is routine. Concerning the existence of $\langle \cdot, \cdot \rangle$ write V as a tensor product of evaluation modules:

$$V = V_{d_1}(a_1) \otimes V_{d_2}(a_2) \otimes \cdots \otimes V_{d_N}(a_N).$$

For $1 \leq i \leq N$ let $\langle \cdot, \cdot \rangle_i$ denote a standard bilinear form on $V_{d_i}(a_i)$. Define a bilinear form $\langle \cdot, \cdot \rangle$ on V such that

$$\langle \otimes_{i=1}^N u_i, \otimes_{i=1}^N v_i \rangle = \prod_{i=1}^N \langle u_i, v_i \rangle_i \quad (59)$$

for all $u_i, v_i \in V_{d_i}(a_i)$ ($1 \leq i \leq N$). By construction \langle, \rangle is nonzero. Using (48) and (59) one checks that \langle, \rangle satisfies (58). To show that \langle, \rangle is nondegenerate, note that the subspace $\{u \in V \mid \langle u, v \rangle = 0 \ \forall v \in V\}$ is a proper \boxtimes -submodule of V and therefore zero by the irreducibility of V . Concerning the uniqueness of \langle, \rangle , let \langle, \rangle' denote any bilinear form on V that satisfies (58). We show that \langle, \rangle' is a scalar multiple of \langle, \rangle . Pick a basis for V , and for $\xi \in \boxtimes$ let ξ_b denote the matrix that represents ξ with respect to this basis. Let M (resp. N) denote the matrix that represents \langle, \rangle (resp. \langle, \rangle') with respect to the basis. Note that M is invertible since \langle, \rangle is nondegenerate. By (58) we have $\xi_b^t M = -M \xi_b$ and $\xi_b^t N = -N \xi_b$ for $\xi \in \boxtimes$. Combining these equations we find that $M^{-1}N$ commutes with ξ_b for all $\xi \in \boxtimes$. Now $M^{-1}N$ is a scalar multiple of the identity by Schur's lemma [8, Lemma 27.3] and since the \boxtimes -module V is irreducible. By these comments N is a scalar multiple of M so \langle, \rangle' is a scalar multiple of \langle, \rangle . We have now shown that \langle, \rangle is unique up to multiplication by a nonzero scalar in \mathbb{F} . To verify our assertions concerning symmetry/asymmetry, let d denote the diameter of V and note that $d = \sum_{i=1}^N d_i$ by Theorem 16.5. Referring to (59),

$$\begin{aligned} \langle \otimes_{i=1}^N u_i, \otimes_{i=1}^N v_i \rangle &= \prod_{i=1}^N \langle u_i, v_i \rangle_i \\ &= \prod_{i=1}^N (-1)^{d_i} \langle v_i, u_i \rangle_i \quad (\text{by Lemma 11.2}) \\ &= (-1)^d \prod_{i=1}^N \langle v_i, u_i \rangle_i \\ &= (-1)^d \langle \otimes_{i=1}^N v_i, \otimes_{i=1}^N u_i \rangle. \end{aligned}$$

It follows that \langle, \rangle is symmetric (resp. antisymmetric) when d is even (resp. d is odd). \square

Before we state our next result we recall a concept. Let V denote a finite-dimensional vector space over \mathbb{F} . By definition the dual space V^* is the vector space over \mathbb{F} consisting of the linear transformations from V to \mathbb{F} . The dimensions of V and V^* coincide. Now assume that V supports a \boxtimes -module structure. Then V^* carries a \boxtimes -module structure such that for $\xi \in \boxtimes$ and $f \in V^*$,

$$(\xi.f)(v) = -f(\xi.v) \quad v \in V. \quad (60)$$

Theorem 18.2 *Let V denote a finite-dimensional irreducible \boxtimes -module and let \langle, \rangle denote a bilinear form on V from Theorem 18.1. Then there exists an isomorphism of \boxtimes -modules $\varphi : V \rightarrow V^*$ such that*

$$\varphi(u)(v) = \langle u, v \rangle \quad u, v \in V. \quad (61)$$

Proof: By elementary linear algebra there exists a unique linear transformation $\varphi : V \rightarrow V^*$ that satisfies (61). The kernel of φ is $\{u \in V \mid \langle u, v \rangle = 0 \ \forall v \in V\}$. This space is zero since \langle, \rangle is nondegenerate, so φ is injective. Now since V, V^* have the same dimension, the map φ is a bijection and hence an isomorphism of vector spaces. Using (58) and (60) one checks that φ is an isomorphism of \boxtimes -modules. \square

19 Finite-dimensional irreducible \boxtimes -modules; more bilinear forms

Let V denote a finite-dimensional irreducible \boxtimes -module. In Theorem 18.1 we displayed a bilinear form \langle , \rangle on V . In this section we mention some related bilinear forms that are of interest.

Theorem 19.1 *Let V denote a finite-dimensional irreducible \boxtimes -module. For a nonidentity $\sigma \in G$ there exists a nonzero bilinear form \langle , \rangle_σ on V such that*

$$\langle \xi.u, v \rangle_\sigma = -\langle u, \sigma(\xi).v \rangle_\sigma \quad \xi \in \boxtimes, \quad u, v \in V. \quad (62)$$

This form is unique up to multiplication by a nonzero scalar in \mathbb{F} . The form is nondegenerate. The form is symmetric.

Proof: Concerning existence, let \langle , \rangle denote a bilinear form on V from Theorem 18.1. Let $\zeta : V \rightarrow V$ denote an isomorphism of \boxtimes -modules from V to V twisted via σ . Define a bilinear form \langle , \rangle_σ on V such that $\langle u, v \rangle_\sigma = \langle u, \zeta(v) \rangle$ for all $u, v \in V$. One checks that \langle , \rangle_σ is nonzero and satisfies (62). Concerning uniqueness, let $\langle , \rangle'_\sigma$ denote a bilinear form on V that satisfies (62). We show that $\langle , \rangle'_\sigma$ is a scalar multiple of \langle , \rangle_σ . Define a bilinear form \langle , \rangle' on V by $\langle u, v \rangle' = \langle u, \zeta^{-1}(v) \rangle'_\sigma$ for all $u, v \in V$. Then \langle , \rangle' satisfies (58). Now by the uniqueness of \langle , \rangle there exists $\beta \in \mathbb{F}$ such that $\langle , \rangle' = \beta \langle , \rangle$. This implies that $\langle , \rangle'_\sigma = \beta \langle , \rangle_\sigma$. The form \langle , \rangle_σ is nondegenerate since the subspace $\{u \in V \mid \langle u, v \rangle_\sigma = 0 \ \forall v \in V\}$ is a proper \boxtimes -submodule of V and therefore zero by the irreducibility of V . We now show that \langle , \rangle_σ is symmetric. Define a bilinear form $\langle , \rangle_\sigma^\dagger$ on V by $\langle u, v \rangle_\sigma^\dagger = \langle v, u \rangle_\sigma$ for all $u, v \in V$. By construction and since $\sigma^2 = 1$ we find $\langle , \rangle_\sigma^\dagger$ satisfies (62). Now by the uniqueness of \langle , \rangle_σ there exists $\gamma \in \mathbb{F}$ such that $\langle , \rangle_\sigma^\dagger = \gamma \langle , \rangle_\sigma$. In other words $\langle v, u \rangle_\sigma = \gamma \langle u, v \rangle_\sigma$ for all $u, v \in V$. We show $\gamma = 1$. By Definition 8.1 there exist mutually distinct $i, j, k, \ell \in \mathbb{I}$ such that $\sigma = (i, j)(k, \ell)$. Let $\{V_n\}_{n=0}^d$ denote the decomposition $[i, j]$ of V , and recall that V_0 has dimension 1 by Theorem 16.5. Setting $\xi = x_{ij}$ in (62) and using $\sigma(x_{ij}) = x_{ji} = -x_{ij}$ we find $\langle x_{ij}.u, v \rangle_\sigma = \langle u, x_{ij}.v \rangle_\sigma$ for all $u, v \in V$. It follows that $\{V_n\}_{n=0}^d$ are mutually orthogonal with respect to \langle , \rangle_σ . By this and since \langle , \rangle_σ is nondegenerate we find the restriction of \langle , \rangle_σ to V_0 is nonzero. By our above comments, for nonzero $u \in V_0$ we have $\langle u, u \rangle_\sigma \neq 0$ and $\langle u, u \rangle_\sigma = \gamma \langle u, u \rangle_\sigma$ so $\gamma = 1$. We conclude that \langle , \rangle_σ is symmetric. \square

At the end of Section 1 we indicated how finite-dimensional irreducible \boxtimes -modules give tridiagonal pairs. The bilinear forms in Theorem 19.1 are useful in the theory of these tridiagonal pairs; we will discuss this connection in a future paper.

20 Suggestions for further research

In this section we give some suggestions for further research. We use the following notation. For an integer $N \geq 1$ let $\mathfrak{sl}_2^{(N)}$ denote the Lie algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2$ (N copies).

Problem 20.1 For an integer $N \geq 1$ and mutually distinct a_1, a_2, \dots, a_N in $\mathbb{F} \setminus \{0, 1\}$, consider the Lie algebra homomorphism $\boxtimes \rightarrow \mathfrak{sl}_2^{(N)}$ that sends $\xi \mapsto (EV_{a_1}(\xi), EV_{a_2}(\xi), \dots, EV_{a_N}(\xi))$ for $\xi \in \boxtimes$. Describe the kernel of this homomorphism in terms of the \boxtimes -generators $\{x_{ij} \mid i, j \in \mathbb{I}, i \neq j\}$ and symmetric functions involving a_1, a_2, \dots, a_N . Also, find an attractive subset of \boxtimes whose image under the homomorphism is a basis for $\mathfrak{sl}_2^{(N)}$.

Example 20.2 With reference to Problem 20.1, assume $N = 2$ and abbreviate $a = a_1$, $b = a_2$. Then the kernel of the homomorphism $\boxtimes \rightarrow \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is generated as an ideal by

$$\begin{aligned} &[x_{12}, x_{03}] - 2ab(x_{23} - x_{01}) - 2(1-a)(1-b)(x_{31} + x_{02}), \\ &[x_{23}, x_{01}] - 2(1-a^{-1})(1-b^{-1})(x_{31} - x_{02}) - 2a^{-1}b^{-1}(x_{12} + x_{03}), \\ &[x_{31}, x_{02}] - 2(1-a)^{-1}(1-b)^{-1}(x_{12} - x_{03}) - 2ab(1-a)^{-1}(1-b)^{-1}(x_{23} + x_{01}). \end{aligned}$$

Also, the images of

$$x_{12}, \quad x_{23}, \quad x_{31}, \quad x_{01}, \quad x_{02}, \quad x_{03}$$

under the homomorphism form a basis for $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

Problem 20.3 Let $U(\boxtimes)$ denote the universal enveloping algebra of \boxtimes . For $\sigma \in G$ find a formal sum

$$\Omega_\sigma = \sum_{n=0}^{\infty} t_n \quad t_n \in U(\boxtimes)$$

such that the following (i)–(iii) hold on each finite-dimensional irreducible \boxtimes -module V :

- (i) t_n is 0 on V for all but finitely many n .
- (ii) Ω_σ is invertible on V .
- (iii) $\sigma(\xi) - \Omega_\sigma \xi \Omega_\sigma^{-1}$ is 0 on V for all $\xi \in \boxtimes$.

Note 20.4 Let V denote a finite-dimensional irreducible \boxtimes -module. Pick $\sigma \in G$ and consider the sum Ω_σ from Problem 20.3. Then the map

$$\begin{aligned} V &\mapsto V \\ v &\mapsto \Omega_\sigma \cdot v \end{aligned}$$

is an isomorphism of \boxtimes -modules from V to V twisted via σ .

Problem 20.5 Find a short direct proof of Theorem 15.2.

Problem 20.6 For an integer $N \geq 1$ let V denote a vector space over \mathbb{F} with dimension $2N$. Let U_i ($i \in \mathbb{I}$) denote N -dimensional subspaces of V such that $U_i \cap U_j = 0$ for distinct $i, j \in \mathbb{I}$. Show that there exists a \boxtimes -module structure on V such that $(x_{ij} + I)U_i = 0$ and $(x_{ij} - I)U_j = 0$ for distinct $i, j \in \mathbb{I}$.

Problem 20.7 With reference to the \boxtimes -module V in Problem 20.6, show that the following are equivalent:

- (i) V is a direct sum of irreducible \boxtimes -modules;
- (ii) there exists a nondegenerate antisymmetric bilinear form $\langle \cdot, \cdot \rangle$ on V such that $\langle U_i, U_i \rangle = 0$ for $i \in \mathbb{I}$.

Problem 20.8 With reference to Notation 12.1, assume \mathbb{F} is the complex number field \mathbb{C} , and that $a \in \mathbb{R}$. Further assume that $\langle \eta_r, \eta_s \rangle \in \mathbb{R}$ for all distinct $r, s \in \mathbb{I}$. Pick mutually distinct $i, j, k, \ell \in \mathbb{I}$ and define $V_{\mathbb{R}} = \sum_{n=0}^d \mathbb{R}u_n$, where $\{u_n\}_{n=0}^d$ is the basis $[i, j, k, \ell]$ of V from Lemma 12.2. By the data in Theorem 13.1 $V_{\mathbb{R}}$ is independent of i, j, k, ℓ . For a nonidentity $\sigma \in G$ consider the restriction of $\langle \cdot, \cdot \rangle_{\sigma}$ to $V_{\mathbb{R}}$. When is this restriction positive definite?

Problem 20.9 Let V denote a finite-dimensional irreducible \boxtimes -module. Find all the linear transformations $\psi : V \rightarrow V$ such that both

$$[x_{01}, [x_{01}, [x_{01}, \psi]]] = 4[x_{01}, \psi], \quad [x_{23}, [x_{23}, [x_{23}, \psi]]] = 4[x_{23}, \psi].$$

Problem 20.10 At the beginning of Section 7 we gave an action of S_4 on \boxtimes . Describe how the S_4 -module \boxtimes decomposes into a direct sum of irreducible S_4 -modules.

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